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**RESEARCH ARTICLE**

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# Weak Hopf algebras corresponding to quantum algebras $U_q(f(K, H))$

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**Abstract** In this paper, we investigate weak Hopf algebras introduced in Li (J Algebra 208:72–100, 1998; Commun Math Phys 225:191–217, 2002) corresponding to quantum algebras $U_q(f(K, H))$ (see Wang et al. in Commun Algebra 30:2191–2211, 2002). A new class of algebras is defined, which is denoted by $\mathfrak{w}U_q^d$. For $d = ((1, 1) \mid (1, 1))$, denote $\mathfrak{w}U_q^d$ briefly by $\mathfrak{w}_1 U_q$; for $d = ((0, 0) \mid (0, 0))$, denote $\mathfrak{w}U_q^d$ briefly by $\mathfrak{w}_2 U_q$. In some cases, the necessary and sufficient conditions for $\mathfrak{w}_1 U_q$ and $\mathfrak{w}_2 U_q$ to be weak Hopf algebras are given. The PBW bases of $\mathfrak{w}_1 U_q$ and $\mathfrak{w}_2 U_q$ are presented. Finally, representations and the center of $\mathfrak{w}_1 U_q$ are characterized over $\mathbb{C}$ with $q \in \mathbb{C}$ which is not a root of unity.

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**الملخص**

نبحث في هذه الورقة جبريات هوبف الضعيفة التي قدمها لي في (مجلة الجبر 208: 72 – 100، 1998؛ تبادل المعلومات في الفيزياء الرياضية 225: 191 – 217، 2002) المناظرة للجبريات الكمية $U_q(f(K,h))$ (انظر وانغ وآخرون في تبادل المعلومات في الجبر 30: 2191 – 2211، 2002). تم تعريف صف جديد من الجبريات يرمز له بالرمز $\mathfrak{w}U_q^d$. عندما $d = ((1,1),(1,1))$ ارمز لـِ $\mathfrak{w}U_q^d$ بشكل موجز بالرمز $\mathfrak{w}_1U_q$؛ وعندما تكون $d = ((0,0),(0,0))$ ارمز لـِ $\mathfrak{w}U_q^d$ بشكل موجز بالرمز $\mathfrak{w}_2U_q$. في بعض الحالات، تعطى شروط كافية وضرورية لتكون $\mathfrak{w}_1U_q$ وَ $\mathfrak{w}_2U_q$ جبريات هوبف ضعيفة. يتم تقديم قواعد PBW لـِ $\mathfrak{w}_1U_q$ وَ $\mathfrak{w}_2U_q$. أخيراً، يتم تمييز تمثيلات ومركز $\mathfrak{w}_1U_q$ على $\mathbb{C}$، حيث $q \in \mathbb{C}$ ليس من جذور 1.

## 1 Introduction

Along with the introduction of quantum groups [6,11], the importance of Hopf algebras [17,20] has been widely recognized in both mathematics and physics. For example, Hopf algebra is a principal tool of studying non-commutative space [16] as quantization of commutative ones. There are some interesting generalizations of Hopf algebras. One way to generalize the concept of Hopf algebra is to introduce a kind of weak co-product such that $\Delta(1) \neq 1 \otimes 1$ (see [4,5]). The paragroups of [18] and the face algebras of [9] are examples of this class of weak Hopf algebras. By defining a weak antipode on bialgebras, Li [13–15] introduced another notion of weak Hopf algebras. The definition is as follows: a $\mathbf{k}$-bialgebra $H = (H, \mu, \eta, \Delta, \varepsilon)$ is called a weak Hopf algebra if there exists $T \in Hom_{\mathbf{k}}(H, H)$ such that $id * T * id = id$ and $T * id * T = T$, where

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$T$ is called a weak antipode of $H$. It should be pointed out that this concept naturally generalizes the notion of Hopf algebras, the left and right Hopf algebras [8]. There are some examples for this kind of weak Hopf algebras. A natural example is the semigroup algebra of any regular monoid which is a natural generalization of group algebra. The weak quantized enveloping algebras of semi-simple Lie algebras, generalized Kac-Moody algebras and super-algebras (see [1,15,22–24]) are also contained in this kind of weak Hopf algebras. The methods to construct such weak Hopf algebras in [1,15,22–24] are similar, that is, replacing the group of group-like elements of the corresponding quantum enveloping algebra by some regular monoid.

Recently, some mathematical workers are interested in the generalization of $U_q(\mathfrak{sl}(2))$ which is the simplest and most important example of quantum groups. As example, Ji and Wang [10] introduced a class of quantum algebras $U_q(f(K))$ and studied representations of $U_q(f(K))$; Wang et al. [21] introduced another class of quantum algebras $U_q(f(K,H))$ and studied representations and the center of $U_q(f(K,H))$ which is a generalization of Drinfeld double of two Hopf algebras (see [21]). Our aim in this paper is to provide more nontrivial examples for weak Hopf algebras under the Li's meaning. So, using the similar method as in [1,15,22–24], we can introduce a class of $\mathbf{k}_q$-algebras corresponding to $U_q(f(K,H))$. Denote this class of $\mathbf{k}_q$-algebras by $\mathfrak{w}U_q^d(f(K,\overline{K},H,\overline{H}))$. For convenience, denote $\mathfrak{w}U_q^d(f(K,\overline{K},H,\overline{H}))$ by $\mathfrak{w}U_q^d$. It should be pointed out that in [1,15,22–24], the corresponding algebras have only one center idempotent, but here are two center idempotents $P$ and $Q$ in $\mathfrak{w}U_q^d$. If $d=((1,1)\mid(1,1))$, denote $\mathfrak{w}U_q^d$ by $\mathfrak{w}_1U_q$. If $d=((0,0)\mid(0,0))$, denote $\mathfrak{w}U_q^d$ by $\mathfrak{w}_2U_q$. From the definitions of $\mathfrak{w}_1U_q$ and $\mathfrak{w}_2U_q$, we can conclude that $\mathfrak{w}_2U_q$ is a homomorphic image of $\mathfrak{w}_1U_q$ (see Proposition 2.9). We also study the basis of $\mathfrak{w}_1U_q$ and $\mathfrak{w}_2U_q$. They are Noetherian but with zero divisors. If we add the relation $P+Q=1$, we can find that there are invertible elements in $\mathfrak{w}U_q^d$ and $\mathfrak{w}_1U_q$ can be decomposed into two ideals which are isomorphic to two algebras belonging to the class of quantum algebras $U_q(f(K))$ introduced in [10]. In this condition, the algebra $U_{K,L}{}^{(bialg)22}$ introduced in [7] belongs to this class of $\mathbf{k}_q$-algebras $\mathfrak{w}U_q^d$. If $f(K,\overline{K},H,\overline{H})\neq 0$, we characterize all weak Hopf algebra structures of $\mathfrak{w}_1U_q$ under some assumption. We also give a necessary and sufficient condition for $\mathfrak{w}_2U_q$ to be a weak Hopf algebra. These are weak Hopf algebras but not Hopf algebras. If $P=Q$, these weak Hopf algebras can be written as a direct sum of two ideals and one of them as Hopf algebra is isomorphic to a Hopf algebra belonging to the class of quantum algebras $U_q(f(K,H))$. When $\mathbf{k}$ is the field of complex numbers, $q\in\mathbb{C}$, and $q$ is not a root of unity, we find that the simple modules of $\mathfrak{w}_1U_q$ can be related with that of $U_q(f(K,H))$, $U_q(f(K))$ and $\mathbb{C}\langle E,F\rangle/(EF-FE-b)$ for $b\in\mathbb{C}$. Finally, using the Harish-Chandra homomorphism, we study the center of $\mathfrak{w}_1U_q$. In a word, the algebraic structure and representations of $\mathfrak{w}U_q^d$ are related to those of $\mathfrak{w}U_q(\mathfrak{sl}(2))$ (introduced in [15]), $U_q(f(K,H))$, $U_q(f(K))$ and $\mathbb{C}\langle E,F\rangle/(EF-FE-b)$.

The paper is organized as follows: In Sect. 2, we give some notions and the definition of $\mathfrak{w}U_q^d$. We characterize the PBW bases of $\mathfrak{w}_1U_q$ and $\mathfrak{w}_2U_q$. In Sect. 3, we study the weak Hopf algebra structures of $\mathfrak{w}_1U_q$ and $\mathfrak{w}_2U_q$. In some cases, the necessary and sufficient conditions for $\mathfrak{w}_1U_q$ and $\mathfrak{w}_2U_q$ to be weak Hopf algebras are given. In Sect. 4, we investigate representations of $\mathfrak{w}_1U_q$ and describe the center of $\mathfrak{w}_1U_q$. Although the ideals in the proofs of Theorem 3.2 and Theorem 3.4 are ordinary, their computations are complicated. So, we put them in an Appendix.

Throughout this paper, denote by $\mathbf{k}$ a field with characteristic zero and $\mathbb{C}$ the field of complex numbers; $\mathbf{N}$ is the set of natural numbers, i.e., $\mathbf{N}=\{0,1,2,\ldots\}$; $\mathbf{Z}$ is the set of all integers.

## 2 Weak quantum algebra $\mathfrak{w}U_q^d(f(K,\overline{K},H,\overline{H}))$ and its basis

For the reader's convenience, we recall the definition of $U_q(f(K_1,H_1))$ (see [21]). Let $\mathbf{k}$ be an algebraically closed field with characteristic zero and $q$ be an indeterminate. We use $\mathbf{k}_q$ to denote the fraction field of the domain $\mathbf{k}[q,q^{-1}]$. The $\mathbf{k}_q$-algebra $U_q(f(K_1,H_1))$ is generated by six variables $E_1, F_1, K_1, K_1^{-1}, H_1, H_1^{-1}$ with the relations

$$K_1H_1=H_1K_1, \tag{2.1}$$

$$K_1^{-1}K_1=K_1K_1^{-1}=H_1H_1^{-1}=H_1^{-1}H_1=1, \tag{2.2}$$

$$K_1E_1K_1^{-1}=q^2E_1,\quad K_1F_1K_1^{-1}=q^{-2}F_1, \tag{2.3}$$

$$H_1 E_1 H_1^{-1} = q^{-2} E_1, \quad H_1 F_1 H_1^{-1} = q^2 F_1, \tag{2.4}$$

$$[E_1, F_1] = f(K_1, H_1), \tag{2.5}$$

where $f(K_1, H_1) = \sum_{i,j=-N}^{N} a_{ij} K_1^i H_1^j \in \mathbf{k}_q[K_1, K_1^{-1}, H_1, H_1^{-1}]$.

Using the same method as in [15], we weaken the invertibility in (2.2) to regularity. Instead of $\{K_1, K_1^{-1}\}$ and $\{H_1, H_1^{-1}\}$, we introduce two pairs $\{K, \overline{K}\}$ and $\{H, \overline{H}\}$ subjecting to the following relations:

$$K\overline{K}K = K, \quad \overline{K}K\overline{K} = \overline{K}, \quad H\overline{H}H = H, \quad \overline{H}H\overline{H} = \overline{H}. \tag{2.6}$$

So, we can introduce two projectors $P$ and $Q$ such that

$$1 \neq P = K\overline{K} = \overline{K}K, \; PK = KP = K, \quad P\overline{K} = \overline{K}P = \overline{K}, \tag{2.7}$$

$$1 \neq Q = H\overline{H} = \overline{H}H, \; QH = HQ = H, \quad Q\overline{H} = \overline{H}Q = \overline{H}. \tag{2.8}$$

Here, corresponding to the relation (2.1), we also let

$$KH = HK, \quad K\overline{H} = \overline{H}K, \overline{K}H = H\overline{K}, \quad \overline{K}\,\overline{H} = \overline{H}\,\overline{K}. \tag{2.9}$$

To generalize the other relations in the definition, we need some terminologies for simplicity. For example, if $E$ satisfies

$$KE = q^2 EK, \quad \overline{K}E = q^{-2} E\overline{K}, \; HE = q^{-2} EH, \quad \overline{H}E = q^2 E\overline{H}, \tag{2.10}$$

then we consider $E$ to be of type $(1, 1)$. If $E$ satisfies

$$KE = q^2 EK, \quad \overline{K}E = q^{-2} E\overline{K}, \quad HE\overline{H} = q^{-2} E, \tag{2.11}$$

then we consider $E$ to be of type $(1, 0)$. If $E$ satisfies

$$KE\overline{K} = q^2 E, \quad HE = q^{-2} EH, \quad \overline{H}E = q^2 E\overline{H}, \tag{2.12}$$

then we consider $E$ to be of type $(0, 1)$. If $E$ satisfies

$$KE\overline{K} = q^2 E, \quad HE\overline{H} = q^{-2} E, \tag{2.13}$$

then we consider $E$ to be of type $(0, 0)$. The same convention holds for $F$ by replacing $E$ by $F$ and 2 by $-2$, $-2$ by 2 in the relations above.

By (2.7)–(2.8), we can get two formulas as follows:

$$P^{(ij)} = K^i \overline{K}^j = \begin{cases} K^{i-j}, & i > j, \\ P, & i = j \neq 0, \\ 1, & i = j = 0, \\ \overline{K}^{j-i}, & i < j. \end{cases}, \quad Q^{(ij)} = H^i \overline{H}^j = \begin{cases} H^{i-j}, & i > j, \\ Q, & i = j \neq 0, \\ 1, & i = j = 0, \\ \overline{H}^{j-i}, & i < j. \end{cases}$$

Obviously, both $P^{(ij)}$ and $Q^{(ij)}$ satisfy the condition of regularity

$$P^{(ij)} P^{(ji)} P^{(ij)} = P^{(ij)}, \quad Q^{(ij)} Q^{(ji)} Q^{(ij)} = Q^{(ij)}. \tag{2.14}$$

**Proposition 2.1** (i) *$E$ (resp. $F$) is of type $(0, 0)$ if and only if $E$ (resp. $F$) is of type $(1, 1)$ and $PE = EP = E$, $QE = EQ = E$, (respectively, $PF = FP = F$, $QF = FQ = F$);*

(ii) *$E$ (resp. $F$) is of type $(0, 1)$ if and only if $E$ (resp. $F$) is of type $(1, 1)$ and $PE = EP = E$, (respectively, $PF = FP = F$);*

(iii) *$E$ (resp. $F$) is of type $(1, 0)$ if and only if $E$ (resp. $F$) is of type $(1, 1)$ and $QE = EQ = E$, (respectively, $QF = FQ = F$).*

*Proof* The proof is similar to that of Proposition 3.1 in [22]. □

For convenience, we write $d = ((d_1, d_2) \mid (\overline{d_1}, \overline{d_2}))$ to denote the types of $E$ and $F$ simultaneously. Here, $d_i, \overline{d_i} \in \{0, 1\}$ for $i = 1, 2$. Having this notion, we say that $E$ and $F$ are of type $d$ in an obvious sense.

Let

$$M(K, \overline{K}, H, \overline{H}) = \Bigg\{ cPQ + \sum_{i,j\geq 0} a_{ij} K^i H^j + \sum_{i\geq 0} b_i Q K^i + \sum_{j\geq 0} c_j P H^j + \sum_{m>0} d_m Q \overline{K}^m + \sum_{m>0} e_m P \overline{H}^m + \sum_{i\geq 0, n>0} h_{in} K^i \overline{H}^n + \sum_{m>0, j\geq 0} g_{mj} \overline{K}^m H^j + \sum_{m>0, n>0} k_{mn} \overline{K}^m \overline{H}^n \Bigg\}.$$

Here, all coefficients are in $\mathbf{k}_q$.

Now, we introduce the definition of $\mathbf{k}_q$-algebras corresponding to $U_q(f(K, H))$ as follows and will give the weak Hopf structures in the sequel.

**Definition 2.2** The $\mathbf{k}_q$-algebra $\mathfrak{w}U_q^d(f(K, \overline{K}, H, \overline{H}))$ is generated by $E, F, K, \overline{K}, H, \overline{H}, P, Q$ with the relations (2.7), (2.8), (2.9), $E, F$ are of type $d$, and

$$[E, F] = f(K, \overline{K}, H, \overline{H}), \tag{2.15}$$

where $f(K, \overline{K}, H, \overline{H}) = \sum_{i,j,m,n=l}^{t} a_{ijmn} K^i \overline{K}^j H^m \overline{H}^n \in M(K, \overline{K}, H, \overline{H})$. Here, $l$ and $t$ are two non-negative integers.

*Remark 2.3* It should be pointed out that the algebra $U_{K,L}{}^{(bialg)22}$ introduced in [7] belongs to the class of $\mathbf{k}_q$-algebras in Definition 2.2 when we substitute $L$ for $\overline{H}$, $\overline{L}$ for $H$, and $f(K, \overline{K}, H, \overline{H}) = \frac{(K+\overline{H})-(\overline{K}+H)}{q-q^{-1}}$ and add the relation $P + Q = 1$. The purpose of [7] is to obtain a new quantum algebra by adding a von Neumann regular Cartan-like generator and some extra relations into $U_q(\mathfrak{sl}(2))$, which admits an embedding of $U_q(\mathfrak{sl}(2))$. Thus, $L$ in $U_{K,L}{}^{(bialg)22}$ is not a group-like element of $U_{K,L}{}^{(bialg)22}$ in [7]. So, the weak Hopf structure of $U_{K,L}{}^{(bialg)22}$ is different from that of $\mathfrak{w}U_q^d(f(K, \overline{K}, H, \overline{H}))$ which will be introduced later.

The following lemma is easy to be checked:

**Lemma 2.4** *$P$ and $Q$ are two center idempotent elements of $\mathfrak{w}U_q^d(f(K, \overline{K}, H, \overline{H}))$.*

For convenience, from now on, denote $\mathfrak{w}U_q^d(f(K, \overline{K}, H, \overline{H}))$ by $\mathfrak{w}U_q^d$.

**Proposition 2.5** (i) *$\mathfrak{w}U_q^d = \mathfrak{w}U_q^d P \oplus \mathfrak{w}U_q^d(1 - P)$, $\mathfrak{w}U_q^d = \mathfrak{w}U_q^d Q \oplus \mathfrak{w}U_q^d(1 - Q)$. Moreover, $\mathfrak{w}U_q^d = \mathfrak{w}U_q^d PQ \oplus \mathfrak{w}U_q^d P(1 - Q) \oplus \mathfrak{w}U_q^d(1 - P)Q \oplus \mathfrak{w}U_q^d(1 - P)(1 - Q)$. Here all sums mean the direct sums of algebras.*

(ii) *$\mathfrak{w}U_q^d/(1 - P, 1 - Q)$ is isomorphic to the algebra $U_q(g(K, H))$, where $g(K, H) = \sum_{i,j,m,n=l}^{t} a_{ijmn} K^{i-j} H^{m-n}$ where $l$ and $t$ are two non-negative integers.*

*Proof* For (i), we only prove $\mathfrak{w}U_q^d = \mathfrak{w}U_q^d P \oplus \mathfrak{w}U_q^d(1 - P)$. The other decompositions can be similarly obtained. For any $x \in \mathfrak{w}U_q^d$, we have $x = xP + x(1 - P)$. Let $y \in \mathfrak{w}U_q^d P \bigcap \mathfrak{w}U_q^d(1 - P)$. Then, $y = y_1 P = y_2(1 - P)$ for some $y_1, y_2 \in \mathfrak{w}U_q^d$. By Lemma 2.4, $P$ is a center idempotent element. Then, we obtain that $y(1 - P) = y_1 P(1 - P) = 0 = y_2(1 - P) = y$. Therefore, we have $\mathfrak{w}U_q^d = \mathfrak{w}U_q^d P \oplus \mathfrak{w}U_q^d(1 - P)$. Since $P$ and $1 - P$ are in the center, the direct sum is the direct sum of algebras.

For (ii), denote the image of any $x \in \mathfrak{w}U_q^d$ in $\mathfrak{w}U_q^d/(1 - P, 1 - Q)$ by $\widetilde{x}$. Then, $\widetilde{P} = \widetilde{Q} = \widetilde{1}$. The morphism of algebras from $\mathfrak{w}U_q^d/(1 - P, 1 - Q)$ to $U_q(g(K, H))$ is given as follows:

$$\widetilde{K} \longmapsto K, \quad \widetilde{\overline{K}} \longmapsto K^{-1}, \quad \widetilde{P} \longmapsto 1, \quad \widetilde{H} \longmapsto H, \quad \widetilde{\overline{H}} \longmapsto H^{-1},$$
$$\widetilde{Q} \longmapsto 1, \quad \widetilde{E} \longmapsto E, \quad \widetilde{F} \longmapsto F.$$

Obviously, it is an isomorphism. □

By Proposition 2.1, we can get the lemma as follows:

**Lemma 2.6** *For $m \geq 0, n \geq 0$, the following relations hold in $\mathfrak{w}U_q^d$:*

$$\begin{aligned} E^m K^n &= q^{-2mn} K^n E^m, \; E^m \overline{K}^n = q^{2mn} \overline{K}^n E^m,\\ E^m H^n &= q^{2mn} H^n E^m, \; E^m \overline{H}^n = q^{-2mn} \overline{H}^n E^m,\\ F^m K^n &= q^{2mn} K^n F^m, \; F^m \overline{K}^n = q^{-2mn} \overline{K}^n F^m,\\ F^m H^n &= q^{-2mn} H^n F^m, \; F^m \overline{H}^n = q^{2mn} \overline{H}^n F^m. \end{aligned}$$

In order to compute $[E, F^k]$ and $[E^k, F]$, we recall some notations in [21]. For any element $h(K, \overline{K}, H, \overline{H}) = \sum_{i,j,m,n=l}^{t} a_{ijmn} K^i \overline{K}^j H^m \overline{H}^n$ in $M(K, \overline{K}, H, \overline{H})$ and $l, t, s, k \in \mathbf{N}$ and $k \geq 1$, denote by

$$\begin{aligned} h^{+(s)}(K, \overline{K}, H, \overline{H}) &= \sum_{i,j,m,n=l}^{t} q^{2s(i-j-m+n)} a_{ijmn} K^i \overline{K}^j H^m \overline{H}^n,\\ h^{-(s)}(K, \overline{K}, H, \overline{H}) &= \sum_{i,j,m,n=l}^{t} q^{-2s(i-j-m+n)} a_{ijmn} K^i \overline{K}^j H^m \overline{H}^n,\\ h_{+(k)}(K, \overline{K}, H, \overline{H}) &= \sum_{i,j,m,n=l}^{t} (k)_{q^{2(i-j-m+n)}} a_{ijmn} K^i \overline{K}^j H^m \overline{H}^n,\\ h_{-(k)}(K, \overline{K}, H, \overline{H}) &= \sum_{i,j,m,n=l}^{t} (k)_{q^{-2(i-j-m+n)}} a_{ijmn} K^i \overline{K}^j H^m \overline{H}^n. \end{aligned}$$

Here $(k)_{q^i} = 1 + q^i + q^{2i} + \cdots + q^{(k-1)i}$ for any $i \in \mathbf{Z}$.
Using these notations, we can obtain the lemma as follows:

**Lemma 2.7** *Let $s, k \in \mathbf{N}$ and $k \geq 1$. For any element $h(K, \overline{K}, H, \overline{H}) = \sum_{i,j,m,n=l}^{t} a_{ijmn} K^i \overline{K}^j H^m \overline{H}^n$ in $M(K, \overline{K}, H, \overline{H})$, the following relations hold:*

(1)

$$\begin{aligned} h(K, \overline{K}, H, \overline{H}) F^s &= F^s h^{-(s)}(K, \overline{K}, H, \overline{H}), \quad h(K, \overline{K}, H, \overline{H}) E^s = E^s h^{+(s)}(K, \overline{K}, H, \overline{H}),\\ h(K, \overline{K}, H, \overline{H}) E^s &= E^s h^{+(s)}(K, \overline{K}, H, \overline{H}), \quad h(K, \overline{K}, H, \overline{H}) E^s = E^s h^{+(s)}(K, \overline{K}, H, \overline{H}),\\ F^s h(K, \overline{K}, H, \overline{H}) &= h^{+(s)}(K, \overline{K}, H, \overline{H}) F^s, \quad E^s h(K, \overline{K}, H, \overline{H}) = h^{-(s)}(K, \overline{K}, H, \overline{H}) E^s. \end{aligned}$$

(2)

$$\begin{aligned} h_{+(k)}(K, \overline{K}, H, \overline{H}) &= \sum_{s=0}^{k-1} h^{+(s)}(K, \overline{K}, H, \overline{H}),\\ h_{-(k)}(K, \overline{K}, H, \overline{H}) &= \sum_{s=0}^{k-1} h^{-(s)}(K, \overline{K}, H, \overline{H}). \end{aligned}$$

**Lemma 2.8** *For $k \geq 1$, the following relations hold in $\mathfrak{w}U_q^d$:*

$$[E, F^k] = F^{k-1} f_{-(k)}(K, \overline{K}, H, \overline{H}) = f_{+(k)}(K, \overline{K}, H, \overline{H}) F^{k-1},$$

$$[E^k, F] = E^{k-1} f_{+(k)}(K, \overline{K}, H, \overline{H}) = f_{-(k)}(K, \overline{K}, H, \overline{H}) E^{k-1}.$$

*Proof* This lemma can be easily obtained by induction on $k$ and using (2.15). □

From now on, we mainly consider the two extreme cases: $d = ((1,1)|(1,1))$ and $d = ((0,0)|(0,0))$. When $d = ((1,1)|(1,1))$, for convenience, denote $\mathfrak{w}U_q^d$ by $\mathfrak{w}_1 U_q$. For $d = ((0,0)|(0,0))$, denote $\mathfrak{w}U_q^d$ by $\mathfrak{w}_2 U_q$.

**Proposition 2.9**

$$\mathfrak{w}_1 U_q/(PE-E, QE-E, PF-F, QF-F) \cong \mathfrak{w}_2 U_q.$$

*Proof* By Proposition 2.1, $E$ (resp. $F$) is of type $(0,0)$ if and only if $E$ (resp. $F$) is of type $(1,1)$ and $PE = EP = E$, $QE = EQ = E$ (resp. $PF = FP = F$, $QF = FQ = F$). Therefore, we can give a morphism of algebras from $\mathfrak{w}_1 U_q$ to $\mathfrak{w}_2 U_q$. For avoiding confusion in notations, let the generators of $\mathfrak{w}_i U_q$ be $K_i, \overline{K}_i, H_i, \overline{H}_i, P_i, Q_i, E_i, F_i$ for $i \in \{1,2\}$. Then, the morphism of algebras is just as follows: $X_1 \longmapsto X_2$ for any $X \in \{K, \overline{K}, H, \overline{H}, P, Q, E, F\}$. Obviously, it is surjective. $P_1E_1 - E_1$, $Q_1E_1 - E_1$, $P_1F_1 - F_1$ and $Q_1F_1 - F_1$ are in the kernel of this morphism. Since $P_1$ and $Q_1$ are in the center, by Proposition 2.1, the kernel of this morphism is just the ideal generated by $P_1E_1 - E_1$, $Q_1E_1 - E_1$, $P_1F_1 - F_1$ and $Q_1F_1 - F_1$. Therefore, this proposition holds. □

Next, using weak Ore extension (see in [15]), we study the basis of $\mathfrak{w}_1 U_q$.

**Theorem 2.10** *The algebra $\mathfrak{w}_1 U_q$ is Noetherian with the basis $A = \{E^lF^kK^iH^j, E^lF^kQK^i, E^lF^kPH^j, E^lF^kQ\overline{K}^m, E^lF^kP\overline{H}^n, E^lF^kK^i\overline{H}^n, E^lF^k\overline{K}^mH^j, E^lF^k \cdot \overline{K}^m\overline{H}^n, E^lF^kPQ\}$, where $i, j, l, k$ are any non-negative integers, $m$ and $n$ are any positive integers.*

*Proof* The proof is similar to that of Theorem 2 in [15] and also is referred to that of Proposition 2.5 in [21]. □

Next, we give the similar theorem for $\mathfrak{w}_2 U_q$.

**Theorem 2.11** *The algebra $\mathfrak{w}_2 U_q$ is Noetherian with the basis $B = \{E^lF^kK^iH^j, QK^i, PH^j, Q\overline{K}^m, P\overline{H}^n, E^lF^kK^i\overline{H}^n, E^lF^k\overline{K}^mH^j, E^lF^k\overline{K}^m\overline{H}^n, PQ\}$, where $i, j, l, k$ are any non-negative integers, $m$ and $n$ are any positive integers.*

*Proof* By Proposition 2.9 and Theorem 2.10, $\mathfrak{w}_2 U_q$ is Noetherian.

Next, we use the Diamond Lemma (see [2, Theorem 1.2]) to prove this theorem.

Now, we define a semigroup partial ordering $\leq$ on the set $W$ of words in the generators $E, F, K, \overline{K}, H, \overline{H}, P$ and $Q$. First, define the lexicographic ordering $\leq_{lex}$ on $W$ by ordering the generators as $E, P, Q, F, K, \overline{K}, H, \overline{H}$. For each word $w \in W$, let $l(w)$ be the total number of times $E$ and $F$ appearing in $w$. Now, given two words $w$ and $v$, we define $w \leq v$ if

(1) $l(w) < l(v)$, or
(2) $l(w) = l(v)$ and $\text{length}(w) < \text{length}(v)$, or
(3) $l(w) = l(v)$ and $\text{length}(w) = \text{length}(v)$, $w \leq_{lex} v$.

This is a semigroup partial ordering which satisfies the descending chain condition and is also compatible with the reduction system given by the relations in Definition 2.2 (later, we will write down our reduction system). We have to check that the ambiguities are resolvable, which we will do below. Then, the Diamond Lemma implies that the irreducible words $\{E^lF^kK^iH^j, QK^i, PH^j, Q\overline{K}^m, P\overline{H}^n, E^lF^kK^i\overline{H}^n, E^lF^k\overline{K}^mH^j, E^lF^k\overline{K}^m\overline{H}^n, PQ\}$ form a basis for $\mathfrak{w}_2 U_q$, where $i, j, l, k$ are any non-negative integers, $m$ and $n$ are any positive integers.

Here, we give the details of the verification. Let us write down our reduction system:

$$\begin{aligned}
&PE \longmapsto E, \quad EP \longmapsto E, \quad QE \longmapsto E, \quad EQ \longmapsto E,\\
&FE \longmapsto EF - f(K, \overline{K}, H, \overline{H}), \quad KE \longmapsto q^2EK, \quad \overline{K}E \longmapsto q^{-2}E\overline{K},\\
&HE \longmapsto q^{-2}EH, \quad \overline{H}E \longmapsto q^2E\overline{H}, \quad QP \longmapsto PQ, \quad FP \longmapsto F, \quad PF \longmapsto F,\\
&KP \longmapsto K, \quad PK \longmapsto K, \quad \overline{K}P \longmapsto \overline{K}, \quad P\overline{K} \longmapsto \overline{K}, \quad HP \longmapsto PH, \quad \overline{H}P \longmapsto P\overline{H},\\
&FQ \longmapsto F, \quad QF \longmapsto F, \quad KQ \longmapsto QK, \quad \overline{K}Q \longmapsto Q\overline{K},\\
&HQ \longmapsto H, \quad QH \longmapsto H, \quad \overline{H}Q \longmapsto \overline{H}, \quad Q\overline{H} \longmapsto \overline{H},\\
&KF \longmapsto q^{-2}FK, \quad \overline{K}F \longmapsto q^2F\overline{K}, \quad HF \longmapsto q^2FH, \quad \overline{H}F \longmapsto q^{-2}F\overline{H},\\
&\overline{K}K \longmapsto P, \quad K\overline{K} \longmapsto P, \quad HK \longmapsto KH, \quad \overline{H}K \longmapsto K\overline{H},\\
&H\overline{K} \longmapsto \overline{K}H, \quad \overline{H}\overline{K} \longmapsto \overline{K}\overline{H}, \quad \overline{H}H \longmapsto Q, \quad H\overline{H} \longmapsto Q.
\end{aligned}$$

By the reduction system above, we conclude that there is no inclusion ambiguity and all overlap ambiguities appear in words of length 3. Thus, the words which we have to check are

$$\begin{array}{llllllllll} QPE, & FPE, & KPE, & \overline{K}PE, & HPE, & \overline{H}PE, & FQE, & KQE, & \overline{K}QE, & \\ HQE, & \overline{H}QE, & KFE, & \overline{K}FE, & HFE, & \overline{H}FE, & \overline{K}KE, & HKE, & \overline{H}KE, & \\ H\overline{K}E, & \overline{H}\overline{K}E, & \overline{H}HE, & FQP, & KQP, & \overline{K}QP, & HQP, & \overline{H}QP, & KFP, & \\ \overline{K}FP, & HFP, & \overline{H}FP, & \overline{K}KP, & HKP, & \overline{H}KP, & H\overline{K}P, & \overline{H}\overline{K}P, & \overline{H}HP, & KFQ, \\ \overline{K}FQ, & HFQ, & \overline{H}FQ, & \overline{K}KQ, & HKQ, & \overline{H}KQ, & H\overline{K}Q, & \overline{H}\overline{K}Q, & \overline{H}HQ, & \\ \overline{K}KF, & HKF, & \overline{H}KF, & H\overline{K}F, & \overline{H}\overline{K}F, & \overline{H}HF, & H\overline{K}K, & \overline{H}\overline{K}K, & \overline{H}HK, & \overline{H}H\overline{K}. \end{array}$$

We now show that all these ambiguities are resolvable:

$$\begin{aligned} &(QP)E \longmapsto P(QE) \longmapsto PE \longmapsto E, \\ &Q(PE) \longmapsto QE \longmapsto E, \\ &(FP)E \longmapsto FE \longmapsto EF - f(K, \overline{K}, H, \overline{H}), \\ &F(PE) \longmapsto FE \longmapsto EF - f(K, \overline{K}, H, \overline{H}), \\ &(KP)E \longmapsto KE \longmapsto q^2EK, \\ &K(PE) \longmapsto KE \longmapsto q^2EK. \end{aligned}$$

As above, we can easily check that the ambiguities in words $\overline{K}PE$, $HPE$, $\overline{H}PE$, $FQE$, $KQE$, $\overline{K}QE$, $HQE$ and $\overline{H}QE$ are resolvable.

$$\begin{aligned} (KF)E \longmapsto q^{-2}F(KE) \longmapsto (FE)K &\longmapsto (EF - f(K, \overline{K}, H, \overline{H}))K \\ &\longmapsto EFK - f(K, \overline{K}, H, \overline{H})K, \\ K(FE) \longmapsto K(EF - f(K, \overline{K}, H, \overline{H})) &\longmapsto q^2E(KF) - Kf(K, \overline{K}, H, \overline{H}) \\ &\longmapsto EFK - Kf(K, \overline{K}, H, \overline{H}). \end{aligned}$$

Since $f(K, \overline{K}, H, \overline{H}) \in M(K, \overline{K}, H, \overline{H})$, it is easy to check that the ambiguity in $KFE$ is resolvable. Similarly, we also can show that the ambiguities in $\overline{K}FE$, $HFE$, $\overline{H}FE$, $\overline{K}KE$ are resolvable.

Moreover, it is easy to check that the other ambiguities are resolvable. □

*Remark 2.12* Here, for any $d$, $\mathfrak{w}U_q^d$ has zero divisors. For example, $(1 - P)$ and $(1 - Q)$ are two zero divisors. In particular, for $\mathfrak{w}_2U_q$, $(1 - P)(1 - Q)$ can annihilate all generators.

Specially, with the relation $P + Q = 1$ in Definition 2.2, we can get the following results about $\mathfrak{w}U_q^d$:

**Proposition 2.13** *$aK^m + bH^n$, $aK^m + b\overline{H}^n$, $a\overline{K}^m + bH^n$, and $a\overline{K}^m + b\overline{H}^n$ are invertible and their inverses are $a^{-1}\overline{K}^m + b^{-1}\overline{H}^n$, $a^{-1}\overline{K}^m + b^{-1}H^n$, $a^{-1}K^m + b^{-1}\overline{H}^n$ and $a^{-1}K^m + b^{-1}H^n$, respectively, where $a, b \in \mathbf{k}_q\backslash\{0\}$, $m$ and $n$ are positive integers.*

*Proof* Since $P + Q = 1$, we have $PQ = QP = (1 - P)P = 0$. By (2.7) and (2.8), we obtain $KH = KPQH = 0$. Similarly, we get

$$KH = \overline{K}H = \overline{KH} = K\overline{H} = 0.$$

Therefore, $(aK^m + bH^n)(a^{-1}\overline{K}^m + b^{-1}\overline{H}^n) = P + Q = 1$. The other cases can be proved similarly. □

If $P + Q = 1$, we have the Pierce decomposition: $\mathfrak{w}_1U_q = P\mathfrak{w}_1U_q \oplus Q\mathfrak{w}_1U_q$. Then, we can get the proposition as follows:

**Proposition 2.14** *As algebras, $\mathfrak{w}_1U_q = U_q(g(K_1)) \oplus U_q(p(K_1))$, where $g(K) = g(K, K^{-1})$ the sum of all terms of $f(K, \overline{K}, H, \overline{H})$ in which the degrees of $H$ and $\overline{H}$ are $0$, while $p(H) = p(H, H^{-1})$ the sum of all terms of $f(K, \overline{K}, H, \overline{H})$ in which the degrees of $K$ and $\overline{K}$ are $0$.*

*Proof* The isomorphism is given as follows:

$$\begin{aligned} &K \longmapsto K_1 \oplus 0, \quad \overline{K} \longmapsto K_1^{-1} \oplus 0, \quad P \longmapsto 1 \oplus 0, \quad PE \longmapsto E_1 \oplus 0, \quad PF \longmapsto F_1 \oplus 0, \\ &H \longmapsto 0 \oplus K_1^{-1}, \quad \overline{H} \longmapsto 0 \oplus K_1, \quad Q \longmapsto 0 \oplus 1, \quad QE \longmapsto 0 \oplus E_1, \quad QF \longmapsto 0 \oplus F_1. \end{aligned}$$

□

*Remark on Proposition 2.14* The definition of quantum algebras $U_q(g(K_1))$ or $U_q(p(K_1))$ can be referred to [10].

## 3 Weak Hopf algebra structure of $\mathfrak{w}U_q^d$

In this section, we mainly consider weak Hopf algebra structures on $\mathfrak{w}_1U_q$ and $\mathfrak{w}_2U_q$.

First, we investigate bialgebra structures on $\mathfrak{w}_1U_q$ and $\mathfrak{w}_2U_q$. Let $\Delta$ be a map from $\mathfrak{w}U_q^d$ into $\mathfrak{w}U_q^d\otimes\mathfrak{w}U_q^d$ such that

$$\Delta(K)=K\otimes K,\quad \Delta(\overline{K})=\overline{K}\otimes\overline{K},\quad \Delta(H)=H\otimes H,\quad \Delta(\overline{H})=\overline{H}\otimes\overline{H}, \tag{3.1}$$
$$\Delta(E)=K^{i_1}\overline{K}^{j_1}H^{m_1}\overline{H}^{n_1}\otimes E+E\otimes K^{i_2}\overline{K}^{j_2}H^{m_2}\overline{H}^{n_2}, \tag{3.2}$$
$$\Delta(F)=K^{p_1}\overline{K}^{q_1}H^{s_1}\overline{H}^{t_1}\otimes F+F\otimes K^{p_2}\overline{K}^{q_2}H^{s_2}\overline{H}^{t_2}, \tag{3.3}$$

where $K^{i_1}\overline{K}^{j_1}H^{m_1}\overline{H}^{n_1}$, $K^{i_2}\overline{K}^{j_2}H^{m_2}\overline{H}^{n_2}$, $K^{p_1}\overline{K}^{q_1}H^{s_1}\overline{H}^{t_1}$, and $K^{p_2}\overline{K}^{q_2}H^{s_2}\overline{H}^{t_2}$ belong to $M(K,\overline{K},H,\overline{H})$, and $i_1,i_2,j_1,j_2,m_1,m_2,n_1,n_2,p_1,p_2,q_1,q_2,s_1,s_2,t_1,t_2\in\mathbf{N}$.

With the equalities above, to make a bialgebra structure of $\mathfrak{w}U_q^d$, we define a map $\varepsilon$ from $\mathfrak{w}U_q^d$ to $\mathbf{k}_q$ as follows:

$$\varepsilon(K)=\varepsilon(\overline{K})=1,\quad \varepsilon(H)=\varepsilon(\overline{H})=1,\quad \varepsilon(E)=\varepsilon(F)=0. \tag{3.4}$$

Then, the following lemma holds:

**Lemma 3.1** $\mathfrak{w}_1U_q$ *is a bialgebra with comultiplication* $\Delta$ *and counit* $\varepsilon$ *defined above if and only if* $(m_2-n_2)+(j_2-i_2)=(t_1-s_1)+(p_1-q_1)$, $(m_1-n_1)+(j_1-i_1)=(t_2-s_2)+(p_2-q_2)$, *and*

$$f(K,\overline{K},H,\overline{H})=a(K^{i_1+p_1}\overline{K}^{j_1+q_1}H^{m_1+s_1}\overline{H}^{n_1+t_1}-K^{i_2+p_2}\overline{K}^{j_2+q_2}H^{m_2+s_2}\overline{H}^{n_2+t_2}).$$

*Here,* $a\in\mathbf{k}_q$.

*Similarly,* $\mathfrak{w}_2U_q$ *is a bialgebra with comultiplication* $\Delta$ *and counit* $\varepsilon$ *defined above if and only if* $(m_2-n_2)+(j_2-i_2)=(t_1-s_1)+(p_1-q_1)$, $(m_1-n_1)+(j_1-i_1)=(t_2-s_2)+(p_2-q_2)$, *and*

$$f(K,\overline{K},H,\overline{H})=a(K^{i_1+p_1}\overline{K}^{j_1+q_1}H^{m_1+s_1}\overline{H}^{n_1+t_1}-K^{i_2+p_2}\overline{K}^{j_2+q_2}H^{m_2+s_2}\overline{H}^{n_2+t_2}),$$

*where* $x_u+y_u\neq 0$ *for any* $(x,y)\in\{(i,j),(m,n),(p,q),(s,t)\}$ *and any* $u\in\{1,2\}$. *Here,* $a\in\mathbf{k}_q$.

*Proof* The proof is similar to that of Lemma 3.2 in [21] or that of Proposition 3.2 in [10]. □

*Remark* In Lemma 3.1, we use $x_u+y_u\neq 0$ just to say equivalently that $x_u$ and $y_u$ cannot be equal to 0 simultaneously for any $(x,y)\in\{(i,j),(m,n),(p,q),(s,t)\}$ and any $u\in\{1,2\}$. Since all the numbers are non-negative integers, here, this notation is more brief and will not bring any confusion. In the sequel, for convenience, we often use this description.

Next, we consider weak Hopf algebra structures of $\mathfrak{w}_1U_q$ and $\mathfrak{w}_2U_q$. To make the bialgebra structure in Lemma 3.1 into a weak Hopf algebra structure, we should define a map $T$ from $\mathfrak{w}U_q^d$ to $\mathfrak{w}U_q^d$ as follows:

$$T(K)=\overline{K},\qquad T(\overline{K})=K,\qquad T(H)=\overline{H},\qquad T(\overline{H})=H, \tag{3.5}$$
$$T(E)=bK^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1}E,\qquad T(F)=cK^{u_2}\overline{K}^{v_2}H^{r_2}\overline{H}^{w_2}F, \tag{3.6}$$

where $K^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1}$ and $K^{u_2}\overline{K}^{v_2}H^{r_2}\overline{H}^{w_2}$ belong to $M(K,\overline{K},H,\overline{H})$, $b,c\in\mathbf{k}_q$, and $u_1,u_2,v_1,v_2,r_1,r_2,w_1,w_2\in\mathbf{N}$. Moreover, we can define $T$ to be an anti-algebra morphism of $\mathfrak{w}U_q^d$.

For convenience, we first write down 26 kinds of weak Hopf structures of $\mathfrak{w}_1U_q$ as follows (here, we just give the equalities of $\Delta(E)$, $\Delta(F)$, $T(E)$, $T(F)$, $f(K,\overline{K},H,\overline{H})$ concretely. The other equalities are the same as those in Lemma 3.1):

**Case 1**

$$\Delta(E) = K^{i_1}\overline{K}^{j_1}H^{m_1}\overline{H}^{n_1} \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes K^{p_2}\overline{K}^{q_2}H^{s_2}\overline{H}^{t_2},$$
$$T(E) = -K^{j_1}\overline{K}^{i_1}H^{n_1}\overline{H}^{m_1}E, \quad T(F) = -q^{-2t_2+2s_2+2q_2-2p_2}K^{q_2}\overline{K}^{p_2}H^{t_2}\overline{H}^{s_2}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(K^{i_1}\overline{K}^{j_1}H^{m_1}\overline{H}^{n_1} - K^{p_2}\overline{K}^{q_2}H^{s_2}\overline{H}^{t_2}),$$

where $i_1, j_1, m_1, n_1, p_2, q_2, s_2$ and $t_2$ are non-negative integers, $i_1 + j_1 \neq 0, m_1 + n_1 \neq 0, p_2 + q_2 \neq 0, s_2 + t_2 \neq 0, (m_1 - n_1) + (j_1 - i_1) = (t_2 - s_2) + (p_2 - q_2)$ and $a \in \mathbf{k}_q \backslash \{0\}$.

**Case 2**

$$\Delta(E) = 1 \otimes E + E \otimes K^{i_2}\overline{K}^{j_2}H^{m_2}\overline{H}^{n_2}, \quad \Delta(F) = K^{p_1}\overline{K}^{q_1}H^{s_1}\overline{H}^{t_1} \otimes F + F \otimes 1,$$
$$T(E) = -q^{2n_2-2m_2-2j_2+2i_2}K^{j_2}\overline{K}^{i_2}H^{n_2}\overline{H}^{m_2}E, \quad T(F) = -K^{q_1}\overline{K}^{p_1}H^{t_1}\overline{H}^{s_1}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(K^{p_1}\overline{K}^{q_1}H^{s_1}\overline{H}^{t_1} - K^{i_2}\overline{K}^{j_2}H^{m_2}\overline{H}^{n_2}),$$

where $i_2, j_2, m_2, n_2, p_1, q_1, s_1$ and $t_1$ are non-negative integers, $i_2 + j_2 \neq 0, m_2 + n_2 \neq 0, p_1 + q_1 \neq 0, s_1 + t_1 \neq 0, (m_2 - n_2) + (j_2 - i_2) = (t_1 - s_1) + (p_1 - q_1)$ and $a \in \mathbf{k}_q \backslash \{0\}$.

**Case 3**

$$\Delta(E) = PH^{m_1}\overline{H}^{n_1} \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes K^{p_2}\overline{K}^{q_2}H^{s_2}\overline{H}^{t_2},$$
$$T(E) = -H^{n_1}\overline{H}^{m_1}E, \quad T(F) = -q^{-2t_2+2s_2+2q_2-2p_2}K^{q_2}\overline{K}^{p_2}H^{t_2}\overline{H}^{s_2}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(PH^{m_1}\overline{H}^{n_1} - K^{p_2}\overline{K}^{q_2}H^{s_2}\overline{H}^{t_2}),$$

where $m_1, n_1, p_2, q_2, s_2, t_2$ are non-negative integers, $m_1 + n_1 \neq 0, p_2 + q_2 \neq 0, s_2 + t_2 \neq 0, m_1 - n_1 = (t_2 - s_2) + (p_2 - q_2)$ and $a \in \mathbf{k}_q \backslash \{0\}$.

**Case 4**

$$\Delta(E) = PQ \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes K^{p_2}\overline{K}^{q_2}H^{s_2}\overline{H}^{t_2},$$
$$T(E) = -E, \qquad T(F) = -K^{q_2}\overline{K}^{p_2}H^{t_2}\overline{H}^{s_2}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(PQ - K^{p_2}\overline{K}^{q_2}H^{s_2}\overline{H}^{t_2}),$$

where $p_2, q_2, s_2, t_2$ are non-negative integers, $p_2 + q_2 \neq 0, s_2 + t_2 \neq 0, (t_2 - s_2) + (p_2 - q_2) = 0$ and $a \in \mathbf{k}_q \backslash \{0\}$.

**Case 5**

$$\Delta(E) = 1 \otimes E + E \otimes PH^{m_2}\overline{H}^{n_2}, \quad \Delta(F) = K^{p_1}\overline{K}^{q_1}H^{s_1}\overline{H}^{t_1} \otimes F + F \otimes 1,$$
$$T(E) = -q^{2n_2-2m_2}H^{n_2}\overline{H}^{m_2}E, \quad T(F) = -K^{q_1}\overline{K}^{p_1}H^{t_1}\overline{H}^{s_1}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(K^{p_1}\overline{K}^{q_1}H^{s_1}\overline{H}^{t_1} - PH^{m_2}\overline{H}^{n_2}),$$

where $m_2, n_2, p_1, q_1, s_1, t_1$ are non-negative integers, $m_2 + n_2 \neq 0, p_1 + q_1 \neq 0, s_1 + t_1 \neq 0, m_2 - n_2 = (t_1 - s_1) + (p_1 - q_1)$ and $a \in \mathbf{k}_q \backslash \{0\}$.

**Case 6**

$$\Delta(E) = 1 \otimes E + E \otimes PQ, \quad \Delta(F) = K^{p_1}\overline{K}^{q_1}H^{s_1}\overline{H}^{t_1} \otimes F + F \otimes 1,$$
$$T(E) = -E, \quad T(F) = -K^{q_1}\overline{K}^{p_1}H^{t_1}\overline{H}^{s_1}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(K^{p_1}\overline{K}^{q_1}H^{s_1}\overline{H}^{t_1} - PQ),$$

where $p_1, q_1, s_1, t_1$ are non-negative integers, $p_1 + q_1 \neq 0, s_1 + t_1 \neq 0, (t_1 - s_1) + (p_1 - q_1) = 0$ and $a \in \mathbf{k}_q \backslash \{0\}$.

**Case 7**

$$\Delta(E) = K^{i_1}\overline{K}^{j_1}Q \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes K^{p_2}\overline{K}^{q_2}H^{s_2}\overline{H}^{t_2},$$
$$T(E) = -K^{j_1}\overline{K}^{i_1}E, \quad T(F) = -q^{-2t_2+2s_2+2q_2-2p_2}K^{q_2}\overline{K}^{p_2}H^{t_2}\overline{H}^{s_2}F,$$
$$f(K,\overline{K},H,\overline{H}) = a(K^{i_1}\overline{K}^{j_1}Q - K^{p_2}\overline{K}^{q_2}H^{s_2}\overline{H}^{t_2}),$$

where $i_1, j_1, p_2, q_2, s_2, t_2$ are non-negative integers, $i_1 + j_1 \neq 0$, $p_2 + q_2 \neq 0$, $s_2 + t_2 \neq 0$, $j_1 - i_1 = (t_2 - s_2) + (p_2 - q_2)$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 8**

$$\Delta(E) = 1 \otimes E + E \otimes K^{i_2}\overline{K}^{j_2}Q, \quad \Delta(F) = K^{p_1}\overline{K}^{q_1}H^{s_1}\overline{H}^{t_1} \otimes F + F \otimes 1,$$
$$T(E) = -q^{2i_2-2j_2}K^{j_2}\overline{K}^{i_2}E, \quad T(F) = -K^{q_1}\overline{K}^{p_1}H^{t_1}\overline{H}^{s_1}F,$$
$$f(K,\overline{K},H,\overline{H}) = a(K^{p_1}\overline{K}^{q_1}H^{s_1}\overline{H}^{t_1} - K^{i_2}\overline{K}^{j_2}Q),$$

where $i_2, j_2, p_1, q_1, s_1, t_1$ are non-negative integers, $i_2 + j_2 \neq 0$, $p_1 + q_1 \neq 0$, $s_1 + t_1 \neq 0$, $j_2 - i_2 = (t_1 - s_1) + (p_1 - q_1)$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 9**

$$\Delta(E) = PH^{m_1}\overline{H}^{n_1} \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes PH^{s_2}\overline{H}^{t_2},$$
$$T(E) = -H^{n_1}\overline{H}^{m_1}E, \quad T(F) = -q^{-2t_2+2s_2}H^{t_2}\overline{H}^{s_2}F,$$
$$f(K,\overline{K},H,\overline{H}) = a(PH^{m_1}\overline{H}^{n_1} - PH^{s_2}\overline{H}^{t_2}),$$

where $m_1, n_1, s_2, t_2$ are non-negative integers, $m_1 + n_1 \neq 0$, $s_2 + t_2 \neq 0$, $m_1 - n_1 = t_2 - s_2$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 10**

$$\Delta(E) = PH^{m_1}\overline{H}^{n_1} \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes K^{p_2}\overline{K}^{q_2}Q,$$
$$T(E) = -H^{n_1}\overline{H}^{m_1}E, \quad T(F) = -q^{2q_2-2p_2}K^{q_2}\overline{K}^{p_2}F,$$
$$f(K,\overline{K},H,\overline{H}) = a(PH^{m_1}\overline{H}^{n_1} - K^{p_2}\overline{K}^{q_2}Q),$$

where $m_1, n_1, p_2, q_2$ are non-negative integers, $m_1+n_1 \neq 0$, $p_2+q_2 \neq 0$, $m_1-n_1 = p_2-q_2$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 11**

$$\Delta(E) = H^{m_1}\overline{H}^{n_1} \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes H^{s_2}\overline{H}^{t_2},$$
$$T(E) = -H^{n_1}\overline{H}^{m_1}E, \quad T(F) = -q^{-2t_2+2s_2}H^{t_2}\overline{H}^{s_2}F,$$
$$f(K,\overline{K},H,\overline{H}) = a(H^{m_1}\overline{H}^{n_1} - H^{s_2}\overline{H}^{t_2}),$$

where $m_1, n_1, s_2, t_2$ are non-negative integers, $m_1 + n_1 \neq 0$, $s_2 + t_2 \neq 0$, $m_1 - n_1 = t_2 - s_2$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 12**

$$\Delta(E) = 1 \otimes E + E \otimes PH^{m_2}\overline{H}^{n_2}, \quad \Delta(F) = K^{p_1}\overline{K}^{q_1}Q \otimes F + F \otimes 1,$$
$$T(E) = -q^{2n_2-2m_2}H^{n_2}\overline{H}^{m_2}E, \quad T(F) = -K^{q_1}\overline{K}^{p_1}F,$$
$$f(K,\overline{K},H,\overline{H}) = a(K^{p_1}\overline{K}^{q_1}Q - PH^{m_2}\overline{H}^{n_2}),$$

where $m_2, n_2, p_1, q_1$ are non-negative integers, $m_2+n_2 \neq 0$, $p_1+q_1 \neq 0$, $m_2-n_2 = p_1-q_1$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 13**

$$\Delta(E) = K^{i_1}\overline{K}^{j_1}Q \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes K^{p_2}\overline{K}^{q_2}Q,$$
$$T(E) = -K^{j_1}\overline{K}^{i_1}E, \quad T(F) = -q^{2q_2-2p_2}K^{q_2}\overline{K}^{p_2}F,$$
$$f(K,\overline{K},H,\overline{H}) = a(K^{i_1}\overline{K}^{j_1}Q - K^{p_2}\overline{K}^{q_2}Q),$$

where $i_1, j_1, p_2, q_2$ are non-negative integers, $i_1 + j_1 \neq 0$, $p_2 + q_2 \neq 0$, $j_1 - i_1 = p_2 - q_2$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 14**

$$\Delta(E) = 1 \otimes E + E \otimes K^{i_2}\overline{K}^{j_2}, \quad \Delta(F) = K^{p_1}\overline{K}^{q_1} \otimes F + F \otimes 1,$$
$$T(E) = -q^{2i_2-2j_2}K^{j_2}\overline{K}^{i_2}E, \quad T(F) = -K^{q_1}\overline{K}^{p_1}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(K^{p_1}\overline{K}^{q_1} - K^{i_2}\overline{K}^{j_2}),$$

where $i_2, j_2, p_1, q_1$ are non-negative integers, $i_2 + j_2 \neq 0$, $p_1 + q_1 \neq 0$, $j_2 - i_2 = p_1 - q_1$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 15**

$$\Delta(E) = 1 \otimes E + E \otimes K^{i_2}\overline{K}^{j_2}H^{m_2}\overline{H}^{n_2}, \quad \Delta(F) = PH^{s_1}\overline{H}^{t_1} \otimes F + F \otimes 1,$$
$$T(E) = -q^{2n_2-2m_2-2j_2+2i_2}K^{j_2}\overline{K}^{i_2}H^{n_2}\overline{H}^{m_2}E, \quad T(F) = -H^{t_1}\overline{H}^{s_1}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(PH^{s_1}\overline{H}^{t_1} - K^{i_2}\overline{K}^{j_2}H^{m_2}\overline{H}^{n_2}),$$

where $i_2, j_2, m_2, n_2, s_1, t_1$ are non-negative integers, $i_2 + j_2 \neq 0, m_2 + n_2 \neq 0, s_1 + t_1 \neq 0, t_1 - s_1 = (m_2 - n_2) + (j_2 - i_2)$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 16**

$$\Delta(E) = 1 \otimes E + E \otimes K^{i_2}\overline{K}^{j_2}H^{m_2}\overline{H}^{n_2}, \quad \Delta(F) = PQ \otimes F + F \otimes 1,$$
$$T(E) = -K^{j_2}\overline{K}^{i_2}H^{n_2}\overline{H}^{m_2}E, \quad T(F) = -F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(PQ - K^{i_2}\overline{K}^{j_2}H^{m_2}\overline{H}^{n_2}),$$

where $i_2, j_2, m_2, n_2$ are non-negative integers, $i_2 + j_2 \neq 0, m_2 + n_2 \neq 0, (m_2 - n_2) + (j_2 - i_2) = 0$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 17**

$$\Delta(E) = K^{i_1}\overline{K}^{j_1}H^{m_1}\overline{H}^{n_1} \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes PH^{s_2}\overline{H}^{t_2},$$
$$T(E) = -K^{j_1}\overline{K}^{i_1}H^{n_1}\overline{H}^{m_1}E, \quad T(F) = -q^{-2t_2+2s_2}H^{t_2}\overline{H}^{s_2}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(K^{i_1}\overline{K}^{j_1}H^{m_1}\overline{H}^{n_1} - PH^{s_2}\overline{H}^{t_2}),$$

where $i_1, j_1, m_1, n_1, s_2, t_2$ are non-negative integers, $i_1 + j_1 \neq 0, m_1 + n_1 \neq 0, s_2 + t_2 \neq 0, t_2 - s_2 = (m_1 - n_1) + (j_1 - i_1)$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 18**

$$\Delta(E) = K^{i_1}\overline{K}^{j_1}H^{m_1}\overline{H}^{n_1} \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes PQ,$$
$$T(E) = -K^{j_1}\overline{K}^{i_1}H^{n_1}\overline{H}^{m_1}E, \quad T(F) = -F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(K^{i_1}\overline{K}^{j_1}H^{m_1}\overline{H}^{n_1} - PQ),$$

where $i_1, j_1, m_1, n_1$ are non-negative integers, $i_1 + j_1 \neq 0, m_1 + n_1 \neq 0, (m_1 - n_1) + (j_1 - i_1) = 0$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 19**

$$\Delta(E) = 1 \otimes E + E \otimes K^{i_2}\overline{K}^{j_2}H^{m_2}\overline{H}^{n_2}, \quad \Delta(F) = K^{p_1}\overline{K}^{q_1}Q \otimes F + F \otimes 1,$$
$$T(E) = -q^{2n_2-2m_2-2j_2+2i_2}K^{j_2}\overline{K}^{i_2}H^{n_2}\overline{H}^{m_2}E, \quad T(F) = -K^{q_1}\overline{K}^{p_1}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(K^{p_1}\overline{K}^{q_1}Q - K^{i_2}\overline{K}^{j_2}H^{m_2}\overline{H}^{n_2}),$$

where $i_2, j_2, m_2, n_2, p_1, q_1$ are non-negative integers, $i_2 + j_2 \neq 0, m_2 + n_2 \neq 0, p_1 + q_1 \neq 0, (m_2 - n_2) + (j_2 - i_2) = p_1 - q_1$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 20**

$$\Delta(E) = K^{i_1}\overline{K}^{j_1}H^{m_1}\overline{H}^{n_1} \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes K^{p_2}\overline{K}^{q_2}Q,$$
$$T(E) = -K^{j_1}\overline{K}^{i_1}H^{n_1}\overline{H}^{m_1}E, \quad T(F) = -q^{2q_2-2p_2}K^{q_2}\overline{K}^{p_2}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(K^{i_1}\overline{K}^{j_1}H^{m_1}\overline{H}^{n_1} - K^{p_2}\overline{K}^{q_2}Q),$$

where $i_1, j_1, m_1, n_1, p_2, q_2$ are non-negative integers, $i_1 + j_1 \neq 0, m_1 + n_1 \neq 0, p_2 + q_2 \neq 0, (m_1 - n_1) + (j_1 - i_1) = p_2 - q_2$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 21**

$$\Delta(E) = 1 \otimes E + E \otimes PH^{m_2}\overline{H}^{n_2}, \quad \Delta(F) = PH^{s_1}\overline{H}^{t_1} \otimes F + F \otimes 1,$$
$$T(E) = -q^{2n_2-2m_2}H^{n_2}\overline{H}^{m_2}E, \quad T(F) = -H^{t_1}\overline{H}^{s_1}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(PH^{s_1}\overline{H}^{t_1} - PH^{m_2}\overline{H}^{n_2}),$$

where $m_2, n_2, s_1, t_1$ are non-negative integers, $m_2 + n_2 \neq 0, s_1 + t_1 \neq 0, m_2 - n_2 = t_1 - s_1$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 22**

$$\Delta(E) = 1 \otimes E + E \otimes K^{i_2}\overline{K}^{j_2}Q, \quad \Delta(F) = K^{p_1}\overline{K}^{q_1}Q \otimes F + F \otimes 1,$$
$$T(E) = -q^{2i_2-2j_2}K^{j_2}\overline{K}^{i_2}E, \quad T(F) = -K^{q_1}\overline{K}^{p_1}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(K^{p_1}\overline{K}^{q_1}Q - K^{i_2}\overline{K}^{j_2}Q),$$

where $p_1, q_1, i_2, j_2$ are non-negative integers, $p_1 + q_1 \neq 0, i_2 + j_2 \neq 0, j_2 - i_2 = p_1 - q_1$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 23**

$$\Delta(E) = 1 \otimes E + E \otimes K^{i_2}\overline{K}^{j_2}Q, \quad \Delta(F) = PH^{s_1}\overline{H}^{t_1} \otimes F + F \otimes 1,$$
$$T(E) = -q^{2i_2-2j_2}K^{j_2}\overline{K}^{i_2}E, \quad T(F) = -H^{t_1}\overline{H}^{s_1}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(PH^{s_1}\overline{H}^{t_1} - K^{i_2}\overline{K}^{j_2}Q),$$

where $s_1, t_1, i_2, j_2$ are non-negative integers, $s_1 + t_1 \neq 0, i_2 + j_2 \neq 0, j_2 - i_2 = t_1 - s_1$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 24**

$$\Delta(E) = 1 \otimes E + E \otimes H^{m_2}\overline{H}^{n_2}, \quad \Delta(F) = H^{s_1}\overline{H}^{t_1} \otimes F + F \otimes 1,$$
$$T(E) = -q^{2n_2-2m_2}H^{n_2}\overline{H}^{m_2}E, \quad T(F) = -H^{t_1}\overline{H}^{s_1}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(H^{s_1}\overline{H}^{t_1} - H^{m_2}\overline{H}^{n_2}),$$

where $m_2, n_2, s_1, t_1$ are non-negative integers, $m_2 + n_2 \neq 0, s_1 + t_1 \neq 0, m_2 - n_2 = t_1 - s_1$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 25**

$$\Delta(E) = K^{i_1}\overline{K}^{j_1} \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes K^{p_2}\overline{K}^{q_2},$$
$$T(E) = -K^{j_1}\overline{K}^{i_1}E, \quad T(F) = -q^{2q_2-2p_2}K^{q_2}\overline{K}^{p_2}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(K^{i_1}\overline{K}^{j_1} - K^{p_2}\overline{K}^{q_2}),$$

where $i_1, j_1, p_2, q_2$ are non-negative integers, $i_1 + j_1 \neq 0, p_2 + q_2 \neq 0, j_1 - i_1 = p_2 - q_2$ and $a \in \mathbf{k}_q\backslash\{0\}$.

**Case 26**

$$\Delta(E) = K^{i_1}\overline{K}^{j_1}Q \otimes E + E \otimes 1, \quad \Delta(F) = 1 \otimes F + F \otimes PH^{s_2}\overline{H}^{t_2},$$
$$T(E) = -K^{j_1}\overline{K}^{i_1}E, \quad T(F) = -q^{-2t_2+2s_2}H^{t_2}\overline{H}^{s_2}F,$$
$$f(K, \overline{K}, H, \overline{H}) = a(K^{i_1}\overline{K}^{j_1}Q - PH^{s_2}\overline{H}^{t_2}),$$

where $i_1, j_1, s_2, t_2$ are non-negative integers, $i_1 + j_1 \neq 0, s_2 + t_2 \neq 0, j_1 - i_1 = t_2 - s_2$, and $a \in \mathbf{k}_q\backslash\{0\}$.

**Theorem 3.2** *If $f(K, \overline{K}, H, \overline{H}) \neq 0$, then $\mathfrak{w}_1U_q$ is a weak Hopf algebra with the bialgebra structure defined in Lemma 3.1 and weak antipode in the form (3.5) and (3.6) if and only if one of 26 cases above is satisfied. In other words, the 26 cases above are all weak Hopf algebra structures of $\mathfrak{w}_1U_q$ under the assumption.*

*Remark on the proof of Theorem 3.2* To make a weak Hopf algebra structure of $\mathfrak{w}_1 U_q$, we have to prove that $(T * id * T)(x) = T(x)$, $(id * T * id)(x) = x$ for any $x \in \mathfrak{w}_1 U_q$ and $T(F)T(E) - T(E)T(F) = T(f(K, \overline{K}, H, \overline{H}))$ hold. Using Theorem 2.10 and the equalities above, by a fussy discussion, we can obtain the 26 cases above. The explicit proof is referred to Appendix A.1.

*Remark 3.3* From the 26 kinds of weak Hopf algebra structures of $\mathfrak{w}_1 U_q$ above, we conclude that a weak Hopf algebra may have more than one weak antipodes. For example, in Case 3, letting $m_1 = n_1 \neq 0$, then the bialgebra structure of Case 3 is the same with that in Case 4. But the weak antipode in Case 3 is different from that in Case 4. Moreover, we can see that Case 11, Case 14, Case 24 and Case 25 give a class of weak Hopf algebras corresponding to quantum algebras $U_q(f(K))$ introduced in [10].

For a weak Hopf algebra $H$, if $S$ and $T$ are two weak antipodes of $H$, then $S * id * T$ is also a weak antipode of $H$. Denote $S * id * T$ by $S\Diamond T$. Then, the set of all weak antipodes of $H$ forms a semigroup with the multiplication $\Diamond$. From the 26 cases, we see that for the case $\Delta(E) = PQ \otimes E + E \otimes 1$ and $\Delta(F) = 1 \otimes F + F \otimes K^{p_2}\overline{K}^{q_2} H^{s_2}\overline{H}^{t_2}$ with $p_2 \neq q_2$ and $s_2 \neq t_2$, there are four weak antipodes $T_1, T_2, T_3$ and $T_4$ as follows: $T_1(E) = -E$; $T_2(E) = -PE$; $T_3(E) = -QE$; $T_4(E) = -PQE$. The actions of $T_1, T_2, T_3$ and $T_4$ on other generators of $\mathfrak{w}_1 U_q$ are the same. Under the multiplication $\Diamond$, we can get the following relations:

$$(T_i \Diamond T_j)(x) = T_i(x)$$

for any $i, j \in \{1, 2, 3, 4\}$ and any $x \in \{K, \overline{K}, H, \overline{H}, P, Q, E, F\}$.

By the definition, $T_i$ is an anti-algebra morphism of $\mathfrak{w}_1 U_q$ for any $i \in \{1, 2, 3, 4\}$. In this case, we claim that $T_i \Diamond T_j$ is also an anti-algebra morphism of $\mathfrak{w}_1 U_q$ for any $i, j \in \{1, 2, 3, 4\}$. In fact, using induction, we only need to prove that $(T_i \Diamond T_j)(xy) = (T_i \Diamond T_j)(y) \cdot (T_i \Diamond T_j)(x) = T_i(y)T_i(x)$ for any $x, y \in \{K, \overline{K}, H, \overline{H}, P, Q, E, F\}$. Here, we only check $(T_i \Diamond T_j)(EF) = T_i(F)T_i(E)$ as follows:

$$\begin{aligned}
(T_i \Diamond T_j)(EF) &= (T_i * id)(PQ)T_j(EF) + (T_i * id)(PQF)T_j(EK^{p_2}\overline{K}^{q_2} H^{s_2}\overline{H}^{t_2}) \\
&\quad + (T_i * id)(E)T_j(F) + (T_i * id)(EF)T_j(K^{p_2}\overline{K}^{q_2} H^{s_2}\overline{H}^{t_2}) \\
&= PQT_j(F)T_j(E) + PQFT_j(K^{p_2}\overline{K}^{q_2} H^{s_2}\overline{H}^{t_2})T_j(E) + T_i(PQF)PQT_j(E) \\
&\quad + PQET_j(F) + T_i(E)T_j(F) + PQEFT_j(K^{p_2}\overline{K}^{q_2} H^{s_2}\overline{H}^{t_2}) \\
&\quad + PQT_i(F)E + T_i(E)FT_j(K^{p_2}\overline{K}^{q_2} H^{s_2}\overline{H}^{t_2}) + T_i(F)T_i(E)PQ \\
&= T_i(F)T_i(E).
\end{aligned}$$

The other equalities can be similarly checked. Since $T_i \Diamond T_j$ is also an anti-algebra morphism of $\mathfrak{w}_1 U_q$, we can get that $T_i \Diamond T_j = T_i$ for any $i, j \in \{1, 2, 3, 4\}$. So, the set $\{T_1, T_2, T_3, T_4\}$ is closed under the multiplication $\Diamond$. Hence, $\{T_1, T_2, T_3, T_4\}$ is a semigroup with idempotent elements $T_1, T_2, T_3$ and $T_4$ under the multiplication $\Diamond$. This semigroup is a band in the algebraic theory of semigroup. Similarly, for the case $\Delta(E) = PQ \otimes E + E \otimes 1$, and $\Delta(F) = 1 \otimes F + F \otimes PQ$, there are 11 weak antipodes. Under the multiplication $\Diamond$, the set of these 11 weak antipodes forms a band, too.

**Theorem 3.4** *If $f(K, \overline{K}, H, \overline{H}) \neq 0$, $\mathfrak{w}_2 U_q$ is a weak Hopf algebra with the bialgebra structure defined in Lemma 3.1 and weak antipode in the form (3.5) and (3.6) if and only if*

$$\begin{aligned}
T(E) &= -q^{2n_2-2m_2-2j_2+2i_2} K^{j_1+j_2}\overline{K}^{i_1+i_2} H^{n_1+n_2}\overline{H}^{m_1+m_2} E, \\
T(F) &= -q^{-2t_2+2s_2+2q_2-2p_2} K^{q_1+q_2}\overline{K}^{p_1+p_2} H^{t_1+t_2}\overline{H}^{s_1+s_2} F, \\
f(K, \overline{K}, H, \overline{H}) &= a(K^{i_1+p_1}\overline{K}^{j_1+q_1} H^{m_1+s_1}\overline{H}^{n_1+t_1} - K^{i_2+p_2}\overline{K}^{j_2+q_2} H^{m_2+s_2}\overline{H}^{n_2+t_2}),
\end{aligned}$$

*where $(m_2 - n_2) + (j_2 - i_2) = (t_1 - s_1) + (p_1 - q_1)$, $(m_1 - n_1) + (j_1 - i_1) = (t_2 - s_2) + (p_2 - q_2)$, $a \in \mathbf{k}_q \backslash \{0\}$, and $x_u + y_u \neq 0$ for any $(x, y) \in \{(i, j), (m, n), (p, q), (s, t)\}$ and any $u \in \{1, 2\}$.*

*Remark on the proof of Theorem 3.4* The proof is similar to that in Theorem 3.2. But, the corresponding discussion is more easily. For convenience, the explicit proof is referred to Appendix A.2.

Next, we study the relation between the weak Hopf algebras $\mathfrak{w}_i U_q (i = 1, 2)$ and Hopf algebra $U_q(g(K_1, H_1))$ (its Hopf algebra structure is referred to [21]).

**Proposition 3.5** *If $P = Q$, for $\mathfrak{w}_i U_q$, then it follows that $\mathfrak{w}_i U_q = P\mathfrak{w}_i U_q \oplus (1-P)\mathfrak{w}_i U_q$, where $P\mathfrak{w}_i U_q$ and $(1-P)\mathfrak{w}_i U_q$ are two ideals of $\mathfrak{w}_i U_q$. Moreover, $P\mathfrak{w}_i U_q \cong U_q(g(K_1, H_1))$ as Hopf algebras, where $g(K, H) = f(K, K^{-1}, H, H^{-1})$. Here, $i \in \{1, 2\}$.*

*Proof* By Proposition 2.5, $\mathfrak{w}_i U_q = P\mathfrak{w}_i U_q \oplus (1-P)\mathfrak{w}_i U_q$. Then, we only prove that $P\mathfrak{w}_i U_q \cong U_q(g(K_1, H_1))$ as Hopf algebras. Define an algebra morphism $\rho$ from $P\mathfrak{w}_i U_q$ to $U_q(g(K_1, H_1))$ satisfying

$$\rho(PE) = E_1, \quad \rho(PF) = F_1, \quad \rho(K) = K_1, \quad \rho(P) = 1,$$
$$\rho(\overline{K}) = K_1^{-1}, \quad \rho(PH) = H_1, \quad \rho(P\overline{H}) = H_1^{-1}.$$

It is easy to see that $\rho$ is a Hopf algebra isomorphism. □

In the following, we give all group-like elements of $\mathfrak{w}_i U_q (i = 1, 2)$:

**Lemma 3.6** (see [17], Lemma 5.5.1 or [23], Lemma 4.7) *Let $H$ be a bialgebra which contains subspaces $A_0 \subset A_1$ such that*

(1) *$A_0$ is a (unital) subalgebra of $H$ and $A_1$ is a left, and right $A_0$-module;*
(2) *$A_1$ generates $H$ as an algebra, and $1 \in A_0$;*
(3) *$\Delta(A_0) \subseteq A_0 \otimes A_0$ and $\Delta(A_1) \subseteq A_1 \otimes A_0 + A_0 \otimes A_1$.*
*If we set $A_n = (A_1)^n$ for all $n \geq 1$, then $\{A_n\}$ is a co-algebra filtration of $H$ and $H_0 \subseteq A_0$, where $H_0$ is the coradical of $H$.*

**Proposition 3.7** *The set of all group-like elements of $\mathfrak{w}_i U_q (i = 1, 2)$ is $G = \{P^{(ij)} Q^{(mn)} \mid i, j, m, n \in \mathbf{N}\}$, which forms a regular monoid under the multiplication of $\mathfrak{w}_i U_q (i = 1, 2)$.*

*Proof* By Equalities (2.9) and (2.14), $G$ is a regular monoid. Let $A_0 = \mathbf{k}_q G$ and $A_1 = A_0(\mathbf{k}_q E + \mathbf{k}_q F + A_0)A_0$. Here, set $H = \mathfrak{w}_i U_q$. Obviously, $A_0 \subset A_1$ satisfies the hypotheses of Lemma 3.6, and $A_0 \subset H_0$. Hence, $H_0 = A_0$. It is known that all group-like elements are in $H_0$. Therefore, they are in $A_0$. Obviously, the elements in $G$ are group-like. Thus, this proposition holds. □

By Proposition 3.7, we know that $\mathfrak{w}_1 U_q$ and $\mathfrak{w}_2 U_q$ are two pointed weak Hopf algebras with the coradical $\mathbf{k}_q G$.

## 4 Representations and the center of $\mathfrak{w}_1 U_q$

In this section, set $\mathbf{k} = \mathbb{C}$. Let $q \in \mathbb{C}$ and $q$ not a root of unity. By Proposition 2.9, some properties of $\mathfrak{w}_1 U_q$ can be extended to $\mathfrak{w}_2 U_q$. Here, we mainly consider representations of $\mathfrak{w}_1 U_q$ and characterize the center of $\mathfrak{w}_1 U_q$.

Let $V$ be a $\mathfrak{w} U_q^d$-module, and $0 \neq v \in V$. If $Kv = av$ and $\overline{K}v = \overline{a}v$ for $a, \overline{a} \in \mathbb{C}$, by $K\overline{K}K = K$ and $\overline{K}K\overline{K} = \overline{K}$, we conclude that if $a \neq 0$, $\overline{a}$ should be equal to $a^{-1}$ and if $a = 0$, $\overline{a}$ is equal to 0. Similarly, if $Hv = bv$ and $\overline{H}v = \overline{b}v$ for $b, \overline{b} \in \mathbb{C}$, we obtain that if $b \neq 0$, $\overline{b}$ should be equal to $b^{-1}$ and if $b = 0$, $\overline{b}$ is equal to 0. A vector $v \neq 0$ in $V$ is said to be a weight vector of weight $(a, b) \in \mathbb{C} \times \mathbb{C}$ if $Kv = av$, $Hv = bv$. If, in addition, $Ev = 0$, then we call that $v$ is a highest weight vector of weight $(a, b)$. A $\mathfrak{w} U_q^d$-module is called a highest weight module if it is generated by a highest weight vector.

Given a module $V$ over $\mathfrak{w}_1 U_q$. By Proposition 2.5, $V = PQV \oplus P(1-Q)V \oplus (1-P)QV \oplus (1-P)(1-Q)V$ is a direct sum of $\mathfrak{w}_1 U_q$-modules. Then, it is easy to obtain the lemma as follows.

**Lemma 4.1** *Suppose that $V$ is an irreducible representation over $\mathfrak{w}_1 U_q$. Then, we can obtain the following four cases:*
(i) $V = PQV$; (ii) $V = P(1-Q)V$; (iii) $V = (1-P)QV$; (iv) $V = (1-P)(1-Q)V$.

Next, we give a discussion in the four cases as follows:

If $V = PQV$, then $P.v = v$ and $Q.v = v$ for any $v \in V$. Then, $V$ can be viewed as a module over $\mathfrak{w}_1 U_q/(1-P)(1-Q)$. Note that $\mathfrak{w}_1 U_q/(1-P)(1-Q)$ is isomorphic to $U_q(g(K_1, H_1))$ where $g(K, H) = f(K, K^{-1}, H, H^{-1})$. In this case, all finite dimensional simple modules has been studied in [21].

If $V = P(1-Q)V$, then $P.v = v$ and $Q.v = 0$ for any $v \in V$. Then, $Hv = \overline{H}v = 0$. So, $V$ can be seen as a module over $U_q(g(K))$, where $g(K)$ is equal to the sum of all terms in which the degrees of $H$ and $\overline{H}$ are 0 in $f(K, \overline{K}, H, \overline{H})$.

If $V = (1-P)QV$, then $P.v = 0$ and $Q.v = v$ for any $v \in V$. Then, $Kv = \overline{K}v = 0$. Therefore, $V$ can be seen as a module over $U_q(p(H))$, where $p(H)$ is equal to the sum of all terms in which the degrees of $K$ and $\overline{K}$ are 0 in $f(K, \overline{K}, H, \overline{H})$.

For Case (ii) and Case (iii), representations of $U_q(f(K))$ were studied in [10].

If $V = (1-P)(1-Q)V$, then $P.v = 0$ and $Q.v = 0$. Therefore, $Kv = \overline{K}v = 0$ and $Hv = \overline{H}v = 0$. Then, $V$ can be seen as a module of $\mathbb{C}\langle E, F\rangle/(EF - FE - b)$ where $b$ is the constant term of $f(K, \overline{K}, H, \overline{H})$. If $b = 0$, $V$ can be seen as a module of $\mathbb{C}[E, F]$. Representations of $\mathbb{C}[E, F]$ has been studied in [19]. If $b \neq 0$, $\mathbb{C}\langle E, F\rangle/(EF - FE - b)$ is isomorphic to Weyl algebra. Irreducible representations of Weyl algebra have been investigated in [3].

So, by the discussion above, we can get the theorem as follows:

**Theorem 4.2** *Let $q \in \mathbb{C}$ not a root of unity. All irreducible representations of $\mathfrak{w}_1 U_q$ can be obtained respectively in the following four cases:*

(1) *The irreducible representations of $\mathfrak{w}_1 U_q$ satisfying Lemma 4.1* (i) *can be induced from those of $U_q(g(K, H))$ where $g(K, H) = f(K, K^{-1}, H, H^{-1})$;*
(2) *The irreducible representations of $\mathfrak{w}_1 U_q$ satisfying Lemma 4.1* (ii) *can be induced from those of $U_q(g(K))$ where $g(K)$ is equal to the sum of all terms in which the degrees of $H$ and $\overline{H}$ are 0 in $f(K, \overline{K}, H, \overline{H})$;*
(3) *The irreducible representations of $\mathfrak{w}_1 U_q$ satisfying Lemma 4.1* (iii) *can be induced from those of $U_q(p(H))$ where $p(H)$ is equal to the sum of all terms in which the degrees of $K$ and $\overline{K}$ are 0 in $f(K, \overline{K}, H, \overline{H})$;*
(4) *The irreducible representations of $\mathfrak{w}_1 U_q$ satisfying Lemma 4.1* (iv) *can be induced from those of $\mathbb{C}\langle E, F\rangle/(EF - FE - b)$ where $b$ is the constant term of $f(K, \overline{K}, H, \overline{H})$.*

Next, we consider an infinite-dimensional vector space $V(a, b)$ with denumerable basis $\{v_i\}_{i \in \mathbf{N}}$, where $a, b \in \mathbb{C}$. For $p \geq 0$, set

$$Kv_p = aq^{-2p}v_p, \quad \overline{K}v_p = \overline{a}q^{2p}v_p, \quad Hv_p = bq^{2p}v_p, \quad \overline{H}v_p = \overline{b}q^{-2p}v_p, \tag{4.1}$$

$$Ev_{p+1} = f_{-(p+1)}(a, \overline{a}, b, \overline{b})v_p, \quad Fv_p = v_{p+1}, \quad Ev_0 = 0. \tag{4.2}$$

Then, it is easy to obtain the lemma as follows:

**Lemma 4.3** *The relations above define a $\mathfrak{w}_1 U_q$-module structure on $V(a, b)$. $v_0$ generates $V(a, b)$ as a $\mathfrak{w}_1 U_q$-module and is a highest weight vector of weight $(a, b)$. The highest weight module $V(a, b)$ is called a Verma module.*

As in [21], for $f(K, \overline{K}, H, \overline{H}) = \sum_{i,j,m,n=l}^{t} a_{ijmn} K^i \overline{K}^j H^m \overline{H}^n \in M(K, \overline{K}, H, \overline{H})$ where $l$ and $t$ are two non-negative integers, we define

$$f_q^+(K, \overline{K}, H, \overline{H}) = \sum_{i,j,m,n=l}^{t} \frac{1}{q^{2[(i-j)-(m-n)]} - 1} a_{ijmn} K^i \overline{K}^j H^m \overline{H}^n,$$

$$f_q^-(K, \overline{K}, H, \overline{H}) = \sum_{i,j,m,n=l}^{t} \frac{1}{1 - q^{-2[(i-j)-(m-n)]}} a_{ijmn} K^i \overline{K}^j H^m \overline{H}^n.$$

Denote by $Z(\mathfrak{w}_1 U_q)$ the center of $\mathfrak{w}_1 U_q$. First, we introduce two elements of $\mathfrak{w}_1 U_q$. Let

$$C_q^1 = EFP + f_q^+(K, \overline{K}, H, \overline{H})P = FEP + f_q^-(K, \overline{K}, H, \overline{H})P,$$

and

$$C_q^2 = EFQ + f_q^+(K, \overline{K}, H, \overline{H})Q = FEQ + f_q^-(K, \overline{K}, H, \overline{H})Q.$$

It is easy to see that

**Lemma 4.4** *$C_q^1$ and $C_q^2$ belong to $Z(\mathfrak{w}_1 U_q)$.*

By the 26 kinds of weak Hopf algebra structures of $\mathfrak{w}_1 U_q$, in the sequel, we assume that $f(K,\overline{K},H,\overline{H}) = K^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1} - K^{u_2}\overline{K}^{v_2}H^{r_2}\overline{H}^{w_2}$, where $u_1, v_1, r_1, w_1, u_2, v_2, r_2, w_2$ are non-negative integers and $M = (u_1-v_1)-(r_1-w_1) = (r_2-w_2)-(u_2-v_2)$. From the 26 kinds of weak Hopf algebra structures of $\mathfrak{w}_1 U_q$, we see that $u_1+v_1 \neq 0$, $r_1+w_1 \neq 0$, $u_2+v_2 \neq 0$ and $r_2+w_2 \neq 0$ in all cases except Case 11, Case 14, Case 24, Case 25. In Case 11 and Case 24, we get that $u_1 = v_1 = 0$, $r_1+w_1 \neq 0$, $u_2 = v_2 = 0$ and $r_2+w_2 \neq 0$. In Case 14 and Case 25, we have that $u_1+v_1 \neq 0$, $r_1 = w_1 = 0$, $u_2+v_2 \neq 0$ and $r_2 = w_2 = 0$.

So, we only need to study the center of $\mathfrak{w}_1 U_q$ in two cases. In Case 1, let $u_1+v_1 \neq 0$ and $u_2+v_2 \neq 0$; in Case 2, let $r_1+w_1 \neq 0$ and $r_2+w_2 \neq 0$. The methods to study the center of $\mathfrak{w}_1 U_q$ in two cases are similar. Next, we study the center of $\mathfrak{w}_1 U_q$ in detail when $u_1+v_1 \neq 0$ and $u_2+v_2 \neq 0$.

By Proposition 2.5, we have $\mathfrak{w}_1 U_q = W_1 \oplus W_2 \oplus W_3$, where $W_1 = P\mathfrak{w}_1 U_q$, $W_2 = (1-P)Q\mathfrak{w}_1 U_q$, and $W_3 = (1-P)(1-Q)\mathfrak{w}_1 U_q$.

**Lemma 4.5** *$W_3 \subseteq Z(\mathfrak{w}_1 U_q)$.*

*Proof* By Theorem 2.10, $\{E^l F^k(1-P)(1-Q) \mid l \geq 0, k \geq 0\}$ is a basis of $W_3$. Then, we only need to check that $E^l F^k(1-P)(1-Q)$ commutes with the generators $K,\overline{K},H,\overline{H},P,Q,E$ and $F$ under the multiplication of $\mathfrak{w}_1 U_q$ for any non-negative integers $l$ and $k$. It is easy to see that $E^l F^k(1-P)(1-Q)$ commutes with the generators $K,\overline{K},H,\overline{H},P,Q$ under the multiplication of $\mathfrak{w}_1 U_q$. Since

$$\begin{aligned} E(E^l F^k(1-P)(1-Q)) &= E^l(F^{k-1}f_{-(k)}(K,\overline{K},H,\overline{H}) + F^k E)(1-P)(1-Q) \\ &= E^l F^k(1-P)(1-Q)E, \end{aligned}$$

and

$$\begin{aligned} F(E^l F^k(1-P)(1-Q)) &= (E^l F - E^{l-1}f_{+(l)}(K,\overline{K},H,\overline{H}))F^k(1-P)(1-Q) \\ &= E^l F^k(1-P)(1-Q)F, \end{aligned}$$

the result follows. □

Let $W_i^K$ be the sub-algebra of $W_i$ consisting of all elements commuting with $K$ for $i = 1,2,3$. For any $x \in W_i^K$, $xK = Kx$ and $xP = Px$. It follows that

$$\overline{K}x = \overline{K}Px = \overline{K}xP = \overline{K}xK\overline{K} = Px\overline{K} = x\overline{K}.$$

For convenience, let

$$\begin{aligned} M_1(K,\overline{K},H,\overline{H}) = \Bigg\{ & cPQ + \sum_{i>0,j\geq 0} a_{ij}K^i H^j + \sum_{i>0} b_i QK^i + \sum_{j\geq 0} c_j PH^j + \sum_{m>0} d_m Q\overline{K}^m \\ & + \sum_{m>0} e_m P\overline{H}^m + \sum_{i>0,n>0} h_{in}K^i\overline{H}^n + \sum_{m>0,j\geq 0} g_{mj}\overline{K}^m H^j + \sum_{m>0,n>0} k_{mn}\overline{K}^m\overline{H}^n \Bigg\}. \end{aligned}$$

Then, we give the lemma as follows:

**Lemma 4.6** *Any element of $W_1$ belongs to $W_1^K$ if and only if it is of the form $\sum_{i\geq 0} F^i P_i E^i$ where $P_0, P_1, \ldots$ are elements of $M_1(K,\overline{K},H,\overline{H})$.*

*Proof* The proof is similar to that of Lemma VI.4.2 in [12]. □

Since $\mathfrak{w}_1 U_q = W_1 \oplus W_2 \oplus W_3$, for any $z \in Z(\mathfrak{w}_1 U_q)$, there exists $x \in W_1$, $y \in W_2$, $v \in W_3$ such that $z = x+y+v$. By Lemma 4.5, $v$ belongs to $Z(\mathfrak{w}_1 U_q)$. Therefore, $x+y \in Z(\mathfrak{w}_1 U_q)$. So, $K(x+y) = (x+y)K$. Since $Ky = yK = 0$, we have $Kx = xK$. Therefore, $x \in W_1^K$. By the lemma above, we obtain that $HW_1^K = W_1^K H$. So, $Hx = xH$. Because of $x+y \in Z(\mathfrak{w}_1 U_q)$, we have $H(x+y) = (x+y)H$. Therefore, $Hy = yH$. So, $y \in W_2^H$. Similar to Lemma 4.6, for any element $v$ in $W_2^H$, $v$ is of the form $\sum_{l\geq 0} a_l E^l F^l(1-P)Q + \sum_{l\geq 0,m>0} b_{lm}E^l F^l H^m(1-P) + \sum_{l\geq 0,m>0} c_{lm}E^l F^l\overline{H}^m(1-P)$.

**Lemma 4.7** *$W_2^H$ belongs to $Z(\mathfrak{w}_1 U_q)$.*

*Proof* The proof is similar to that in Lemma 4.5. □

By Lemma 4.7, $x$ belongs to $Z(\mathfrak{w}_1 U_q)$. Therefore, we only need to characterize the center of $W_1$. Similar to studying the center of $U_q(\mathfrak{sl}(2))$ in [12], we also use the Harish-Chandra homomorphism to study the center of $W_1$.

Let $I = W_1 E \cap W_1^K$. Then, it is a left ideal of $W_1^K$.

**Lemma 4.8** *We have $I = FW_1 \cap W_1^K$ and $W_1 = I \oplus M_1(K, \overline{K}, H, \overline{H})$.*

*Proof* The proof is similar to that of Lemma VI.4.3 in [12]. □

Therefore, $I$ is a two-sided ideal of $W_1^K$ and the projection $\varphi$ from $W_1^K$ onto $M_1(K, \overline{K}, H, \overline{H})$ is a morphism of algebras. $\varphi$ is called the Harish-Chandra homomorphism. It permits one to express the action of the center $Z(W_1)$ on a highest weight module.

**Lemma 4.9** *Let $V$ be a highest weight $W_1$-module with highest weight $(a, b)$. Then, for any central element $z \in W_1$ and any $v \in V$, it follows that $zv = \varphi(z)(a, \overline{a}, b, \overline{b})v$, where $\varphi(z)$ belongs to $M_1(K, \overline{K}, H, \overline{H})$ and $\varphi(z)(a, \overline{a}, b, \overline{b})$ is its value at $a, \overline{a}, b, \overline{b}$.*

*Proof* The proof is similar to that of Proposition VI.4.4 in [12]. □

When $u_1+v_1 \neq 0$ and $u_2+v_2 \neq 0$, we have $C_q^1 = EFP + f_q^+(K, \overline{K}, H, \overline{H}) = FEP + f_q^-(K, \overline{K}, H, \overline{H})$. Thus,

$$\varphi(C_q^1) = \frac{1}{q^M - q^{-M}}(q^M K^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1} + q^{-M}K^{u_2}\overline{K}^{v_2}H^{r_2}\overline{H}^{w_2}).$$

Therefore, $C_q^1$ acts on a highest weight module of highest weight $(a, b)$ as the multiplication by the scalar $\frac{1}{q^M - q^{-M}}(q^M a^{u_1}\overline{a}^{v_1}b^{r_1}\overline{b}^{w_1} + q^{-M}a^{u_2}\overline{a}^{v_2}b^{r_2}\overline{b}^{w_2})$.

Next, we prove that the restriction of $\varphi$ to the center $Z(W_1)$ is injective.

**Lemma 4.10** *Let $z \in Z(W_1)$. If $\varphi(z) = 0$, then $z = 0$.*

*Proof* The proof is similar to that of Lemma 5.5 in [21]. □

For any element $s$ in $M_1(K, \overline{K}, H, \overline{H})$, denote by $\widetilde{s}$ the polynomial defined by the change of variable $\widetilde{s}(a, \overline{a}, b, \overline{b}) = s(q^{-1}a, q\overline{a}, qb, q^{-1}\overline{b})$.

**Lemma 4.11** *For any element $z \in Z(W_1)$, it follows that*

$$\widetilde{(\varphi(z))}(\xi a x, \xi^{-1}\overline{a}x^{-1}, \xi^{-1}\overline{a}y, \xi a y^{-1}) = \widetilde{(\varphi(z))}(\xi\overline{a}x, \xi^{-1}ax^{-1}, \xi^{-1}ay, \xi\overline{a}y^{-1})$$

*for $a \in \mathbb{C}$, any $\xi$ with $\xi^{2M} = 1$, and any $x, y$ with $x^{(u_1-u_2)-(v_1-v_2)}y^{(r_1-r_2)-(w_1-w_2)} = 1$.*

*Proof* The proof is similar to that of Lemma 5.6 in [21]. □

**Lemma 4.12** *If $P(K, \overline{K}, H, \overline{H}) \in M_1(K, \overline{K}, H, \overline{H})$ satisfies the relation*

$$P(\xi a, \xi^{-1}\overline{a}, \xi^{-1}\overline{a}, \xi a) = P(\xi\overline{a}, \xi^{-1}a, \xi^{-1}a, \xi\overline{a})$$

*for all $a \in \mathbb{C}$, $M \neq 0$ and $\xi$ with $\xi^{2M} = 1$, then $P(K, \overline{K}, H, \overline{H})$ is generated by $KH$, $\overline{KH}$, $P$, $PQ$, and $K^{b_1}\overline{K}^{c_1}H^{d_1}\overline{H}^{e_1} + K^{b_2}\overline{K}^{c_2}H^{d_2}\overline{H}^{e_2}$ where $b_1, c_1, d_1, e_1, b_2, c_2, d_2, e_2$ are non-negative integers, $b_1 + c_1 \neq 0$, $b_2 + c_2 \neq 0$, and $M = (b_1 - c_1) - (d_1 - e_1) = (d_2 - e_2) - (b_2 - c_2)$.*

*Proof* The proof is similar to that in Lemma 5.7 in [21]. □

**Theorem 4.13** *If $u_1 + v_1 \neq 0$, $u_2 + v_2 \neq 0$ and $M \neq 0$, then the center of $\mathfrak{w}_1 U_q$ is $Z(W_1) \oplus W_2^H \oplus W_3$ where $Z(W_1)$ is a polynomial algebra generated by $KH$, $\overline{KH}$, $P$, $PQ$ and $C_q^1$. The restriction of the Harish-Chandra homomorphism to $Z(W_1)$ is an isomorphism onto the subalgebra of $M_1(K, \overline{K}, H, \overline{H})$ generated by $KH$, $\overline{KH}$, $P$, $PQ$ and $q^M K^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1} + q^{-M}K^{u_2}\overline{K}^{v_2}H^{r_2}\overline{H}^{w_2}$.*

*Proof* By Lemma 4.10, we know that the restriction of $\varphi$ to the center is injective. So, we only need to determine its image. By Lemma 4.11 and Lemma 4.12, the image is contained in the subalgebra of $M_1(K, \overline{K}, H, \overline{H})$ generated by $KH, \overline{KH}, P, PQ$ and $K^{b_1}\overline{K}^{c_1}H^{d_1}\overline{H}^{e_1} + K^{b_2}\overline{K}^{c_2}H^{d_2}\overline{H}^{e_2}$ for $b_1, c_1, d_1, e_1, b_2, c_2, d_2, e_2$ are non-negative integers, $b_1+c_1 \neq 0$, $b_2+c_2 \neq 0$, and $M = (b_1-c_1)-(d_1-e_1) = (d_2-e_2)-(b_2-c_2)$. If the element $z = \sum_{i\geq 1} F^i P_i E^i + q^M K^{b_1}\overline{K}^{c_1}H^{d_1}\overline{H}^{e_1} + q^{-M}K^{b_2}\overline{K}^{c_2}H^{d_2}\overline{H}^{e_2} \in Z(W_1)$, then, by comparing degrees, we can easily get $z = FP_1E + q^M K^{b_1}\overline{K}^{c_1}H^{d_1}\overline{H}^{e_1} + q^{-M}K^{b_2}\overline{K}^{c_2}H^{e_2}\overline{H}^{d_2}$ where $P_1 \in M_1(K, \overline{K}, H, \overline{H})$. Thus, by $Ez = zE$, we get two equalities as follows:

$$FEP_1E = FP_1EE,$$
$$(q^M - q^{-M})(K^{b_1}\overline{K}^{c_1}H^{d_1}\overline{H}^{e_1}E - K^{b_2}\overline{K}^{c_2}H^{d_2}\overline{H}^{e_2}E)$$
$$= K^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1}P_1E - K^{u_2}\overline{K}^{v_2}H^{r_2}\overline{H}^{w_2}P_1E.$$

By the two equalities above, $P_1$ belongs to the sub-algebra generated by $P, PQ, KH$ and $\overline{KH}$. From the 26 cases of $\mathfrak{w}_1U_q$, we know that if $r_1+w_1 = 0$, then $r_2+w_2 = 0$; if $r_1+w_1 \neq 0$, then $r_2+w_2 \neq 0$. If $r_1+w_1 \neq 0$ and $r_2+w_2 \neq 0$, then we get $d_1+e_1 \neq 0$ and $d_2+e_2 \neq 0$. Therefore, if $r_1+w_1 = 0$ and $r_2+w_2 = 0$, it is easy to see that $q^M K^{b_1}\overline{K}^{c_1}H^{d_1}\overline{H}^{e_1} + q^{-M}K^{b_2}\overline{K}^{c_2}H^{d_2}\overline{H}^{e_2}$ is generated by $P, PQ, KH, \overline{KH}$, and $q^M K^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1} + q^{-M}K^{u_2}\overline{K}^{v_2}H^{r_2}\overline{H}^{w_2}$; if $r_1+w_1 \neq 0$ and $r_2+w_2 \neq 0$, it is known that $d_1+e_1 \neq 0$ and $d_2+e_2 \neq 0$, and can be seen that $q^M K^{b_1}\overline{K}^{c_1}H^{d_1}\overline{H}^{e_1} + q^{-M}K^{b_2}\overline{K}^{c_2}H^{d_2}\overline{H}^{e_2}$ is generated by $P, PQ, KH, \overline{KH}$, and $q^M K^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1} + q^{-M}K^{u_2}\overline{K}^{v_2}H^{r_2}\overline{H}^{w_2}$. We have known that

$$\varphi(C_q^1) = \frac{1}{q^M - q^{-M}}(q^M K^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1} + q^{-M}K^{u_2}\overline{K}^{v_2}H^{r_2}\overline{H}^{w_2}).$$

So, the result follows. □

Similarly, we can prove the similar result if $r_1+w_1 \neq 0$ and $r_2+w_2 \neq 0$. From the 26 cases of $\mathfrak{w}_1U_q$, we know that if $u_1+v_1 = 0$, then $u_2+v_2 = 0$; if $u_1+v_1 \neq 0$, then $u_2+v_2 \neq 0$. By Proposition 2.5, we can have $\mathfrak{w}_1U_q = Y_1 \oplus Y_2 \oplus Y_3$, where $Y_1 = Q\mathfrak{w}_1U_q$, $Y_2 = (1-Q)P\mathfrak{w}_1U_q$ and $Y_3 = (1-P)(1-Q)\mathfrak{w}_1U_q$. Then, using the similar method as above, we obtain the theorem as follows:

**Theorem 4.14** *If $r_1+w_1 \neq 0, r_2+w_2 \neq 0$ and $M \neq 0$, then the center of $\mathfrak{w}_1U_q$ is $Z(Y_1) \oplus Y_2^K \oplus Y_3$ where $Z(Y_1)$ is a polynomial algebra generated by $KH, \overline{KH}, PQ, Q$ and $C_q^2$. The restriction of the Harish-Chandra homomorphism to $Z(Y_1)$ is an isomorphism onto the subalgebra of $M_1(K, \overline{K}, H, \overline{H})$ generated by $KH, \overline{KH}, PQ, Q$ and $q^M K^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1} + q^{-M}K^{u_2}\overline{K}^{v_2}H^{r_2}\overline{H}^{w_2}$.*

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## Appendix A: The explicit proofs of Theorem 3.2 and Theorem 3.4

### A.1. The explicit proof of Theorem 3.2

Here, we also let

$$T(E) = bK^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1}E, \quad T(F) = cK^{u_2}\overline{K}^{v_2}H^{r_2}\overline{H}^{w_2}F.$$

Obviously, $T$ keeps the relations (2.7)–(2.9) and $E, F$ are of type $d = ((1,1) \mid (1,1))$ in Definition 2.2. It is easy to check that $(T * id * T)(x) = T(x)$ and $(id * T * id)(x) = x$ for any $x \in \{K, \overline{K}, H, \overline{H}, P, Q\}$.

Denote $\Delta(E)$ by $A_1 \otimes E + E \otimes B_1$, and $\Delta(F)$ by $A_2 \otimes F + F \otimes B_2$ still. Then, we calculate $(T * id * T)(E)$ as follows:

$$\begin{aligned}(T * id * T)(E) &= (T * id)(A_1)T(E) + (T * id)(E)T(B_1)\\ &= T(A_1)(A_1)T(E) + T(A_1)ET(B_1) + T(E)B_1T(B_1)\\ &= P^{i_1+j_1}Q^{n_1+m_1}T(E) + q^{2[(n_2-m_2)-(j_2-i_2)]}H^{n_1+n_2}\overline{H}^{m_1+m_2}\\ &\quad \cdot K^{j_1+j_2}\overline{K}^{i_1+i_2}E + P^{i_2+j_2}Q^{m_2+n_2}T(E).\end{aligned}$$

If we want to get that $(T * id * T)(E) = T(E)$, then the following equality should be satisfied:

$$\begin{aligned}&bP^{i_1+j_1}Q^{n_1+m_1}K^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1}E + q^{2[(n_2-m_2)-(j_2-i_2)]}H^{n_1+n_2}\overline{H}^{m_1+m_2}\\ &\cdot K^{j_1+j_2}\overline{K}^{i_1+i_2}E + bP^{i_2+j_2}Q^{m_2+n_2}K^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1}E = bK^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1}E. \qquad \text{(A.1.1)}\end{aligned}$$

Next, we discuss it in three cases.

If $u_1 + v_1 \neq 0$ and $r_1 + w_1 \neq 0$, then the equality above can be written as

$$bK^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1}E + q^{2[(n_2-m_2)-(j_2-i_2)]}H^{n_1+n_2}\overline{H}^{m_1+m_2}K^{j_1+j_2}\overline{K}^{i_1+i_2}E = 0.$$

Since $K^{u_1}\overline{K}^{v_1}H^{r_1}\overline{H}^{w_1}$ and $H^{n_1+n_2}\overline{H}^{m_1+m_2}K^{j_1+j_2}\overline{K}^{i_1+i_2}$ belong to $M(K, \overline{K}, H, \overline{H})$ in which any element has the unique expression, we get that $b = -q^{2[(n_2-m_2)-(j_2-i_2)]}$ and let $u_1 = j_1 + j_2, v_1 = i_1 + i_2, r_1 = n_1 + n_2, w_1 = m_1 + m_2$. So, we have $T(E) = -q^{2n_2-2m_2-2j_2+2i_2}K^{j_1+j_2}\overline{K}^{i_1+i_2}H^{n_1+n_2}\overline{H}^{m_1+m_2}E$.

If $u_1 = v_1 = 0$, the equality becomes:

$$\begin{aligned}&bP^{i_1+j_1}Q^{n_1+m_1}H^{r_1}\overline{H}^{w_1}E + q^{2[(n_2-m_2)-(j_2-i_2)]}H^{n_1+n_2}\overline{H}^{m_1+m_2}K^{j_1+j_2}\overline{K}^{i_1+i_2}E\\ &+bP^{i_2+j_2}Q^{m_2+n_2}H^{r_1}\overline{H}^{w_1}E = bH^{r_1}\overline{H}^{w_1}E.\end{aligned}$$

Here, we also discuss it in two cases.

First, we discuss the case when $i_1 + j_1 \neq 0$. If $i_1 + j_1 \neq 0$, by Theorem 2.10, $PQ^{n_1+m_1}H^{r_1} \cdot \overline{H}^{w_1}E$ and $H^{r_1}\overline{H}^{w_1}E$ are linearly independent. Moreover, $H^{n_1+n_2}\overline{H}^{m_1+m_2}K^{j_1+j_2}\overline{K}^{i_1+i_2}E$, $H^{r_1}\overline{H}^{w_1}E$ are also linearly independent. Therefore, we have that $i_2 + j_2 = 0$, i.e. $i_2 = j_2 = 0$. But, when $i_2 = j_2 = 0$, by Theorem 2.10, $PQ^{n_1+m_1}H^{r_1}\overline{H}^{w_1}E$, $H^{n_1+n_2}\overline{H}^{m_1+m_2}K^{j_1}\overline{K}^{i_1}E$ must be linearly dependent. Therefore, we should obtain $b = -q^{2n_2-2m_2}$ and $i_1 = j_1 \neq 0$. If $r_1 + w_1 \neq 0$, we can let $r_1 = n_1 + n_2$ and $w_1 = m_1 + m_2$; if $r_1 = w_1 = 0$, we must have $m_2 = n_2 = 0$, otherwise, $bQ^{m_2+n_2}E$ and $bE$ are linearly independent. Therefore, we have the equality $PQ^{n_1+m_1}E = PH^{n_1}\overline{H}^{m_1}E$. By Theorem 2.10, we get $n_1 = m_1$.

Next, we give a discussion about the case when $i_1 = j_1 = 0$. If $i_2 + j_2 \neq 0$, by Theorem 2.10, we get two equalities: $bQ^{n_1+m_1}H^{r_1}\overline{H}^{w_1}E = bH^{r_1}\overline{H}^{w_1}E$ and $q^{2[(n_2-m_2)-(j_2-i_2)]}H^{n_1+n_2}\overline{H}^{m_1+m_2}K^{j_2}\overline{K}^{i_2}E = -bP^{i_2+j_2}Q^{m_2+n_2}H^{r_1}\overline{H}^{w_1}E$. If $r_1 + w_1 \neq 0$, by the two equalities, we get $i_2 = j_2 \neq 0, T(E) = -q^{2n_2-2m_2}H^{n_1+n_2} \cdot \overline{H}^{m_1+m_2}E$; if $r_1 = w_1 = 0$, we obtain $i_2 = j_2 \neq 0, m_1 = n_1 = 0, m_2 = n_2$ and $T(E) = -E$. If $i_2 + j_2 = 0$, i.e., $i_2 = j_2 = 0$, we also give a discussion in two cases. If $r_1 + w_1 \neq 0$, we have $T(E) = -q^{2n_2-2m_2}H^{n_1+n_2}\overline{H}^{m_1+m_2}E$; if $r_1 = w_1 = 0$, we get $n_1 = m_1 \neq 0, m_2 = n_2 = 0$ and $T(E) = -E$ when $n_1 + m_1 \neq 0$, and $m_2 = n_2$ and $T(E) = -E$ when $m_1 = n_1 = 0$.

If $r_1 = w_1 = 0$, by the discussion similar to that in the second case, we get the results as follows: If $m_1 + n_1 \neq 0$, we obtain that $m_2 = n_2 = 0, m_1 = n_1 \neq 0$, and $T(E) = -q^{2i_2-2j_2}K^{j_1+j_2}\overline{K}^{i_1+i_2}E$ when $u_1 + v_1 \neq 0$, and $m_2 = n_2 = 0, m_1 = n_1 \neq 0, i_2 = j_2 = 0, i_1 = j_1, T(E) = -E$ when $u_1 = v_1 = 0$. If $m_1 = n_1 = 0$ and $m_2 + n_2 \neq 0$, we get that $n_2 = m_2 \neq 0, T(E) = -q^{2i_2-2j_2}K^{j_1+j_2}\overline{K}^{i_1+i_2}E$ when $u_1 + v_1 \neq 0$, and $n_2 = m_2 \neq 0, i_1 = j_1 = 0, i_2 = j_2$ when $u_1 = v_1 = 0$. If $m_1 = n_1 = 0$ and $m_2 = n_2 = 0$, we have that $T(E) = -q^{2i_2-2j_2}K^{j_1+j_2}\overline{K}^{i_1+i_2}E$ when $u_1 + v_1 \neq 0$, and $i_1 = j_1 \neq 0, i_2 = j_2 = 0, T(E) = -E$ or $i_1 = j_1 = 0, i_2 = j_2, T(E) = -E$ when $u_1 = v_1 = 0$.

We consider the equality $(id * T * id)(E) = E$. By $(id * T * id)(E) = E$, we obtain the equality as follows:

$$\begin{aligned}&P^{i_1+j_1}Q^{m_1+n_1}E + K^{i_1}\overline{K}^{j_1}H^{m_1}\overline{H}^{n_1}T(E)K^{i_2}\overline{K}^{j_2}H^{m_2}\overline{H}^{n_2}\\ &+P^{i_2+j_2}Q^{m_2+n_2}E = E. \qquad \text{(A.1.2)}\end{aligned}$$

Next, we also discuss it in three cases as above.

If $u_1 + v_1 \neq 0, r_1 + w_1 \neq 0$, from the discussion above, we have known that $T(E) = -q^{2n_2-2m_2-2j_2+2i_2} K^{j_1+j_2}\overline{K}^{i_1+i_2} H^{n_1+n_2}\overline{H}^{m_1+m_2} E$. Taking it into the equality above, we have $P^{i_1+j_1} Q^{m_1+n_1} E - P^{i_1+i_2+j_1+j_2} Q^{m_1+m_2+n_1+n_2} E + P^{i_2+j_2} Q^{m_2+n_2} E = E$. Then, we discuss it in two cases. If $i_1+j_1 \neq 0$, then we get $i_2 = j_2 = 0, m_2 = n_2 = 0$ and $m_1+n_1 \neq 0$. So, in this case, we have that $T(E) = -K^{j_1}\overline{K}^{i_1} H^{n_1}\overline{H}^{m_1} E$. If $i_1 = j_1 = 0$, we obtain that $i_2 + j_2 \neq 0, m_1 = n_1 = 0$ and $m_2 + n_2 \neq 0$. Therefore, we get that $T(E) = -q^{2n_2-2m_2-2j_2+2i_2} K^{j_2}\overline{K}^{i_2} H^{n_2}\overline{H}^{m_2} E$. In other words, we obtain that

(1) $u_1 + v_1 \neq 0, \quad r_1 + w_1 \neq 0, \quad i_1 + j_1 \neq 0 \Longrightarrow i_2 = j_2 = 0, \quad m_2 = n_2 = 0,$
$m_1 + n_1 \neq 0, \quad T(E) = -K^{j_1}\overline{K}^{i_1} H^{n_1}\overline{H}^{m_1} E;$

(2) $u_1 + v_1 \neq 0, \quad r_1 + w_1 \neq 0, \quad i_1 = j_1 = 0 \quad \Longrightarrow i_2 + j_2 \neq 0, \quad m_1 = n_1 = 0,$
$m_2 + n_2 \neq 0, \quad T(E) = -q^{2n_2-2m_2-2j_2+2i_2} K^{j_2}\overline{K}^{i_2} H^{n_2}\overline{H}^{m_2} E.$

If $u_1 = v_1 = 0$, we also give a discussion in two cases.

First, we discuss the case when $i_1 + j_1 \neq 0$. If $i_1 + j_1 \neq 0$ and $r_1 + w_1 \neq 0$, from the discussion above, we have known that $i_2 = j_2 = 0, i_1 = j_1 \neq 0$ and $T(E) = -q^{2n_2-2m_2} H^{n_1+n_2}\overline{H}^{m_1+m_2} E$. Then, substituting $T(E)$ in Equality (A.1.2) by $-q^{2n_2-2m_2} H^{n_1+n_2}\overline{H}^{m_1+m_2} E$, we get $PQ^{m_1+m_2} E - PQ^{m_1+m_2+n_1+n_2} E + Q^{m_2+n_2} \cdot E = E$. Then, by Theorem 2.10, we obtain that $m_2 = n_2 = 0$ and $m_1 + n_1 \neq 0$. So, in this case, $T(E) = -H^{n_1}\overline{H}^{m_1} E$. If $i_1 + j_1 \neq 0$ and $r_1 + w_1 = 0$, similarly, we can get that $i_2 = j_2 = 0, i_1 = j_1 \neq 0$, $n_2 = m_2 = 0$ and $n_1 = m_1$. Therefore, in this subcase, $T(E) = -E$.

Next, we discuss the second case when $i_1 = j_1 = 0$. If $i_1 = j_1 = 0, i_2 + j_2 \neq 0$ and $r_1 + w_1 \neq 0$, from the discussion above, we have known that $i_2 = j_2 \neq 0, T(E) = -q^{2n_2-2m_2} H^{n_1+n_2}\overline{H}^{m_1+m_2} E$. Then, similarly, replacing $T(E)$ in Equality (A.1.2) by $-q^{2n_2-2m_2} H^{n_1+n_2}\overline{H}^{m_1+m_2} E$, we get $Q^{m_1+n_1} E - PQ^{m_1+m_2+n_1+n_2} E + PQ^{m_2+n_2} E = E$. By Theorem 2.10, we get $m_1 = n_1 = 0$ and $m_2 + n_2 \neq 0$. So, in this case, $T(E) = -q^{2n_2-2m_2} H^{n_2}\overline{H}^{m_2} E$. Similarly, if $i_1 = j_1 = 0, i_2 + j_2 \neq 0$ and $r_1 = w_1 = 0$, we have that $n_1 = m_1 = 0, m_2 = n_2$ and $T(E) = -E$. If $i_1 = j_1 = 0, i_2 = j_2 = 0$ and $r_1 + w_1 \neq 0$, we have known that $T(E) = -q^{2n_2-2m_2} H^{n_1+n_2}\overline{H}^{m_1+m_2} E$. Then, replacing $T(E)$ in Equality (A.1.2) by $-q^{2n_2-2m_2} H^{n_1+n_2}\overline{H}^{m_1+m_2} E$, we obtain that $Q^{m_1+n_1} E - Q^{m_1+m_2+n_1+n_2} E + Q^{m_2+n_2} E = E$. So, by Theorem 2.10, if $m_1+n_1 \neq 0$, then we have $m_2 = n_2 = 0$. In this case, $T(E) = -H^{n_1}\overline{H}^{m_1} E$. If $m_2+n_2 \neq 0$, then $m_1 = n_1 = 0$. In this case, $T(E) = -q^{2n_2-2m_2} H^{n_2}\overline{H}^{m_2} E$. If $i_1 = j_1 = 0, i_2 = j_2 = 0$ and $r_1 = w_1 = 0$, similar to the discussion above, we get that $m_1 = n_1 \neq 0$ and $m_2 = n_2 = 0$ when $m_1 + n_1 \neq 0$, and $m_2 = n_2$ when $m_1 = n_1 = 0$. In these two cases, $T(E) = -E$.

In other words, for the case when $u_1 = v_1 = 0$, we can rewrite the results as follows:

(3) $u_1 = v_1 = 0, \ r_1 + w_1 \neq 0, \ i_1 + j_1 \neq 0 \Longrightarrow i_2 = j_2 = 0, \quad m_2 = n_2 = 0,$
$i_1 = j_1 \neq 0, \quad m_1 + n_1 \neq 0, \quad T(E) = -H^{n_1}\overline{H}^{m_1} E;$

(4) $u_1 = v_1 = 0, \quad r_1 = w_1 = 0, \quad i_1 + j_1 \neq 0 \Longrightarrow i_2 = j_2 = 0, \quad i_1 = j_1 \neq 0,$
$m_2 = n_2 = 0, \ m_1 = n_1, \quad T(E) = -E;$

(5) $u_1 = v_1 = 0, \quad r_1 + w_1 \neq 0, \quad i_1 = j_1 = 0, \quad i_2 + j_2 \neq 0 \Longrightarrow i_2 = j_2 \neq 0,$
$m_2 + n_2 \neq 0, \quad m_1 = n_1 = 0, \quad T(E) = -q^{2n_2-2m_2} H^{n_2}\overline{H}^{m_2} E;$

(6) $u_1 = v_1 = 0, \quad r_1 = w_1 = 0, \quad i_1 = j_1 = 0, \quad i_2 + j_2 \neq 0 \Longrightarrow i_2 = j_2 \neq 0,$
$m_2 = n_2, \quad m_1 = n_1 = 0, \quad T(E) = -E;$

(7) $u_1 = v_1 = 0, \quad r_1 + w_1 \neq 0, \quad i_1 = j_1 = 0, \quad i_2 = j_2 = 0, m_1 + n_1 \neq 0$
$\Longrightarrow m_2 = n_2 = 0, \quad T(E) = -H^{n_1}\overline{H}^{m_1} E;$

(8) $u_1 = v_1 = 0, \quad r_1 + w_1 \neq 0, \quad i_1 = j_1 = 0, \quad i_2 = j_2 = 0, m_2 + n_2 \neq 0$
$\Longrightarrow m_1 = n_1 = 0, \quad T(E) = -q^{2n_2-2m_2} H^{n_2}\overline{H}^{m_2} E;$

(9) $u_1 = v_1 = 0, \quad r_1 = w_1 = 0, \quad i_1 = j_1 = 0, \quad i_2 = j_2 = 0, m_1 + n_1 \neq 0$
$\Longrightarrow m_2 = n_2 = 0, \quad m_1 = n_1 \neq 0, \quad T(E) = -E;$

(10) $u_1 = v_1 = 0, \quad r_1 = w_1 = 0, \quad i_1 = j_1 = 0, \quad i_2 = j_2 = 0, m_1 = n_1 = 0$
$\Longrightarrow m_2 = n_2, \quad T(E) = -E.$

If $r_1 = w_1 = 0$, by the discussion similar to that in the second case, we get the results as follows:

(11) $u_1 + v_1 \neq 0, \ r_1 = w_1 = 0, \ m_1 + n_1 \neq 0 \Longrightarrow i_2 = j_2 = 0, \quad m_2 = n_2 = 0,$
$m_1 = n_1 \neq 0, \quad i_1 + j_1 \neq 0, \quad i_2 = j_2 = 0, \quad T(E) = -K^{j_1}\overline{K}^{i_1} E;$

(12) $u_1 = v_1 = 0, \quad r_1 = w_1 = 0, \quad m_1 + n_1 \neq 0 \Longrightarrow i_2 = j_2 = 0, \quad i_1 = j_1,$
$m_2 = n_2 = 0,\ m_1 = n_1 \neq 0, \quad T(E) = -E;$
(13) $u_1 + v_1 \neq 0, \quad r_1 = w_1 = 0, \quad m_1 = n_1 = 0, \quad m_2 + n_2 \neq 0 \Longrightarrow i_1 = j_1 = 0,$
$i_2 + j_2 \neq 0, \quad m_2 = n_2 \neq 0, \quad m_1 = n_1 = 0, \quad T(E) = -q^{2i_2-2j_2} K^{j_2} \overline{K}^{i_2} E;$
(14) $u_1 = v_1 = 0, \quad r_1 = w_1 = 0, \quad m_1 = n_1 = 0, \quad m_2 + n_2 \neq 0 \Longrightarrow i_1 = j_1 = 0,$
$i_2 = j_2, \quad m_2 = n_2 \neq 0, \quad m_1 = n_1 = 0, \quad T(E) = -E;$
(15) $u_1 + v_1 \neq 0, \quad r_1 = w_1 = 0, \quad m_1 = n_1 = 0, \quad m_2 = n_2 = 0, \quad i_1 + j_1 \neq 0 \Longrightarrow$
$i_2 = j_2 = 0, \quad T(E) = -K^{j_1} \overline{K}^{i_1} E;$
(16) $u_1 + v_1 \neq 0, \quad r_1 = w_1 = 0, \quad m_1 = n_1 = 0, \quad m_2 = n_2 = 0, \quad i_1 = j_1 = 0 \Longrightarrow$
$i_2 + j_2 \neq 0, \quad T(E) = -q^{2i_2-2j_2} K^{j_2} \overline{K}^{i_2} E;$
(17) $u_1 = v_1 = 0, \quad r_1 = w_1 = 0, \quad m_1 = n_1 = 0, \quad m_2 = n_2 = 0, \quad i_1 + j_1 \neq 0$
$\Longrightarrow i_2 = j_2 = 0, \quad i_1 = j_1 \neq 0, \quad T(E) = -E;$
(18) $u_1 = v_1 = 0, \quad r_1 = w_1 = 0, \quad m_1 = n_1 = 0, \quad m_2 = n_2 = 0, i_1 = j_1 = 0$
$\Longrightarrow i_2 = j_2, \quad T(E) = -E.$

For $F$, the discussion is similar. Here, we just write down the results about $F$ as follows:

(19) $u_2 + v_2 \neq 0, \quad r_2 + w_2 \neq 0,\ p_1 + q_1 \neq 0 \Longrightarrow \quad p_2 = q_2 = 0, \quad s_2 = t_2 = 0,$
$s_1 + t_1 \neq 0, \quad T(F) = -K^{q_1} \overline{K}^{p_1} H^{t_1} \overline{H}^{s_1} F;$
(20) $u_2 + v_2 \neq 0, \quad r_2 + w_2 \neq 0,\ p_1 = q_1 = 0 \Longrightarrow \quad p_2 + q_2 \neq 0, \quad s_1 = t_1 = 0,$
$s_2 + t_2 \neq 0, \quad T(F) = -q^{-2t_2+2s_2+2q_2-2p_2} K^{q_2} \overline{K}^{p_2} H^{t_2} \overline{H}^{s_2} F;$
(21) $u_2 = v_2 = 0, \quad r_2 + w_2 \neq 0,\ p_1 + q_1 \neq 0 \Longrightarrow \quad p_2 = q_2 = 0, \quad p_1 = q_1 \neq 0,$
$s_2 = t_2 = 0, \quad s_1 + t_1 \neq 0, \quad T(F) = -H^{t_1} \overline{H}^{s_1} F;$
(22) $u_2 = v_2 = 0, \quad p_1 + q_1 \neq 0, \quad r_2 = w_2 = 0 \Longrightarrow p_2 = q_2 = 0, \quad p_1 = q_1 \neq 0,$
$s_2 = t_2 = 0, \quad s_1 = t_1, \quad T(F) = -F;$
(23) $u_2 = v_2 = 0, \quad p_1 = q_1 = 0, \quad p_2 + q_2 \neq 0, \quad r_2 + w_2 \neq 0 \Longrightarrow p_2 = q_2 \neq 0,$
$s_1 = t_1 = 0, \quad s_2 + t_2 \neq 0, \quad T(F) = -q^{-2t_2+2s_2} H^{t_2} \overline{H}^{s_2} F;$
(24) $u_2 = v_2 = 0, \quad p_1 = q_1 = 0, \quad p_2 + q_2 \neq 0, \quad r_2 = w_2 = 0 \Longrightarrow p_2 = q_2 \neq 0,$
$s_1 = t_1 = 0, \quad s_2 = t_2, \quad T(F) = -F;$
(25) $u_2 = v_2 = 0, \quad p_1 = q_1 = 0, \quad p_2 = q_2 = 0, \quad r_2 + w_2 \neq 0, \quad s_2 = t_2 = 0 \Longrightarrow$
$s_1 + t_1 \neq 0, \quad T(F) = -H^{t_1} \overline{H}^{s_1} F;$
(26) $u_2 = v_2 = 0, \quad p_1 = q_1 = 0, \quad p_2 = q_2 = 0, \quad r_2 + w_2 \neq 0, \quad s_2 + t_2 \neq 0 \Longrightarrow$
$s_1 = t_1 = 0, \quad T(F) = -q^{-2t_2+2s_2} H^{t_2} \overline{H}^{s_2} F;$
(27) $u_2 = v_2 = 0, \quad p_1 = q_1 = 0, \quad p_2 = q_2 = 0, \quad r_2 = w_2 = 0, \quad s_1 + t_1 \neq 0 \Longrightarrow$
$s_2 = t_2 = 0, \quad t_1 = s_1 \neq 0, \quad T(F) = -F;$
(28) $u_2 = v_2 = 0, \quad p_1 = q_1 = 0, \quad p_2 = q_2 = 0, \quad r_2 = w_2 = 0, \quad s_1 = t_1 = 0 \Longrightarrow$
$s_2 = t_2, \quad T(F) = -F;$
(29) $r_2 = w_2 = 0, \quad s_1 + t_1 \neq 0, \quad u_2 + v_2 \neq 0, \Longrightarrow s_2 = t_2 = 0, \quad s_1 = t_1 \neq 0,$
$p_2 = q_2 = 0, \quad p_1 + q_1 \neq 0, \quad T(F) = -K^{q_1} \overline{K}^{p_1} F;$
(30) $r_2 = w_2 = 0, \quad s_1 + t_1 \neq 0, \quad u_2 = v_2 = 0 \Longrightarrow s_2 = t_2 = 0, \quad s_1 = t_1 \neq 0,$
$p_2 = q_2 = 0, \quad p_1 = q_1, \quad T(F) = -F;$
(31) $r_2 = w_2 = 0, \quad s_1 = t_1 = 0, \quad s_2 + t_2 \neq 0, \quad u_2 + v_2 \neq 0 \Longrightarrow s_2 = t_2 \neq 0,$
$p_1 = q_1 = 0, \quad p_2 + q_2 \neq 0, \quad T(F) = -q^{2q_2-2p_2} K^{q_2} \overline{K}^{p_2} F;$
(32) $r_2 = w_2 = 0, \quad s_1 = t_1 = 0, \quad s_2 + t_2 \neq 0, \quad u_2 = v_2 = 0 \Longrightarrow s_2 = t_2 \neq 0,$
$p_1 = q_1 = 0, \quad p_2 = q_2, \quad T(F) = -F;$
(33) $r_2 = w_2 = 0, \quad s_1 = t_1 = 0, \quad s_2 = t_2 = 0, \quad u_2 + v_2 \neq 0, \quad p_2 + q_2 \neq 0 \Longrightarrow$
$p_1 = q_1 = 0, \quad T(F) = -q^{2q_2-2p_2} K^{q_2} \overline{K}^{p_2} F;$
(34) $r_2 = w_2 = 0, \quad s_1 = t_1 = 0, \quad s_2 = t_2 = 0, \quad u_2 + v_2 \neq 0, \quad p_2 = q_2 = 0 \Longrightarrow$
$p_1 + q_1 \neq 0, \quad T(F) = -K^{q_1} \overline{K}^{p_1} F;$
(35) $r_2 = w_2 = 0, \quad s_1 = t_1 = 0, \quad s_2 = t_2 = 0, \quad u_2 = v_2 = 0, \quad p_1 + q_1 \neq 0 \Longrightarrow$
$p_2 = q_2 = 0, \quad p_1 = q_1 \neq 0, \quad T(F) = -F;$
(36) $r_2 = w_2 = 0, \quad s_1 = t_1 = 0, \quad s_2 = t_2 = 0, \quad u_2 = v_2 = 0, \quad p_1 = q_1 = 0 \Longrightarrow$
$p_2 = q_2, \quad T(F) = -F.$

Next, we consider that whether $T(E)$ and $T(F)$ are compatible, i.e., whether $T(F)T(E) - T(E)T(F)$ is equal to $T(f(K, \overline{K}, H, \overline{H}))$.

From the discussion above, we conclude that there are many symmetries during all cases for $E$ and $F$. Obviously, all cases for $E$ are symmetric to those for $F$. The second case for $E$ (resp. $F$) is symmetric to the third case for $E$ (resp. $F$). For example, if $T(E)$ in the first case is compatible with $T(F)$ in the third case, then, due to symmetry, the corresponding $T(E)$ in the third case is compatible with the corresponding $T(F)$ in the first case; if $T(E)$ in the second case is compatible with $T(F)$ in the second case, then the corresponding $T(E)$ in the third case is compatible with the corresponding $T(F)$ in the third case. We also find that $T(E) = -E$ in the second case or in the third case is not compatible with any $T(F)$ in the second case or the third case, if one adds the condition $f(K, \overline{K}, H, \overline{H}) \neq 0$ (obviously, this condition is reasonable). Therefore, by the symmetries, we can largely simplify the computation. By fussy computations, we get the aforementioned 26 cases.

Finally, we have to verify that for any $x \in \mathfrak{w}_1 U_q$, $(id * T * id)(x) = x$ and $(T * id * T)(x) = T(x)$. In all cases, we have to check that if $(id * T * id)(x) = x$, $(T * id * T)(x) = T(x)$, $(id * T * id)(y) = y$, $(T * id * T)(y) = T(y)$, for all $x$ and $y$ being generators $E, F, K, \overline{K}, H, \overline{H}, P, Q$, then

$$(id * T * id)(xy) = xy, \qquad (T * id * T)(xy) = T(xy).$$

If this holds, then the antipode axioms is obvious to hold for any elements by induction.

Here, we only check Case 1. The others are similar.

For Case 1, we get that $(id * T)(E) = (1 - PQ)E$, $(T * id)(E) = 0$, $(id * T)(F) = 0$ and $(T * id)(F) = (1 - PQ)F$. Therefore, $(T * id)(EF) = \sum_{(EF)} T((EF)_{(1)})(EF)_{(2)} = \sum_{(E)(F)} T(F_{(1)})T(E_{(1)})E_{(2)}F_{(2)} = \sum_F T(F_{(1)})(\sum_E T(E_{(1)})E_{(2)})F_{(2)} = 0$. Obviously, it is easy to see that if $x, y \in \{K, \overline{K}, H, \overline{H}, P, Q\}$, then $(id * T * id)(xy) = xy$ and $(T * id * T)(xy) = T(xy)$. First,

$$\begin{aligned}(id * T * id)(KE) &= (id * T)(K^{i_1+1}\overline{K}^{j_1} H^{m_1}\overline{H}^{n_1}) \cdot KE + (id * T)(KE) \cdot K \\ &= KQE + (K^{i_1+1}\overline{K}^{j_1} H^{m_1}\overline{H}^{n_1})T(KE)K + KET(K)K \\ &= KQE - KQE + KE \\ &= KE.\end{aligned}$$

Similarly, we can prove that $(id*T*id)(xy) = xy$ and $(T*id*T)(xy) = T(xy)$ when $x \in \{K, \overline{K}, H, \overline{H}, P, Q\}$ and $y \in \{E, F\}$.

And,

$$\begin{aligned}(id * T * id)(EF) &= A_1A_2(T * id)(EF) + A_1F(T * id)(EB_2) \\ &\quad + EA_2(T * id)(B_1F) + EF(T * id)(B_1B_2) \\ &= EA_2(T * id)(B_1F) + EF(T * id)(B_1B_2) \\ &= E(T * id)(F) + PQEF \\ &= EF.\end{aligned}$$

Similarly, we can check that $(id * T * id)(xy) = xy$ and $(T * id * T)(xy) = T(xy)$ when $x, y \in \{E, F\}$. By the relations in Definition 2.2, the claim is proved.

Now, we have completed the proof.

A.2. The explicit proof of Theorem 3.4

For $\mathfrak{w}_2 U_q$, Equality (A.1.1) becomes that

$$\begin{aligned}&bK^{u_1}\overline{K}^{v_1} H^{r_1}\overline{H}^{w_1} E + q^{2[(n_2-m_2)-(j_2-i_2)]} H^{n_1+n_2} \\ &\cdot \overline{H}^{m_1+m_2} K^{j_1+j_2}\overline{K}^{i_1+i_2} E = 0.\end{aligned}$$

Therefore,

$$T(E) = -q^{2n_2-2m_2-2j_2+2i_2} K^{j_1+j_2}\overline{K}^{i_1+i_2} H^{n_1+n_2}\overline{H}^{m_1+m_2} E.$$

Similarly, we get that

$$T(F) = -q^{-2t_2+2s_2+2q_2-2p_2} K^{q_1+q_2}\overline{K}^{p_1+p_2} H^{t_1+t_2}\overline{H}^{s_1+s_2} F.$$

Next, for $\mathfrak{w}_2 U_q$, Equality (A.1.2) becomes that

$$E + K^{i_1}\overline{K}^{j_1} H^{m_1}\overline{H}^{n_1} T(E) K^{i_2}\overline{K}^{j_2} H^{m_2}\overline{H}^{n_2} = 0.$$

Replacing $T(E)$ by $-q^{2n_2-2m_2-2j_2+2i_2} K^{j_1+j_2}\overline{K}^{i_1+i_2} H^{n_1+n_2}\overline{H}^{m_1+m_2} E$, it is easy to check that $(id * T * id)(E) = E$.

Similarly, we can prove that $(id * T * id)(F) = F$.

Then, we verity that $T(F)T(E) - T(E)T(F) = T(f(K, \overline{K}, H, \overline{H}))$.

$$\begin{aligned}
T(F)T(E) - T(E)T(F) &= -q^{-2t_2+2s_2+2q_2-2p_2} K^{q_1+q_2}\overline{K}^{p_1+p_2} H^{t_1+t_2}\overline{H}^{s_1+s_2} F \\
&\quad \cdot(-q^{2n_2-2m_2-2j_2+2i_2} K^{j_1+j_2}\overline{K}^{i_1+i_2} H^{n_1+n_2}\overline{H}^{m_1+m_2} E) \\
&\quad -(-q^{2n_2-2m_2-2j_2+2i_2} K^{j_1+j_2}\overline{K}^{i_1+i_2} H^{n_1+n_2}\overline{H}^{m_1+m_2} E) \\
&\quad \cdot(-q^{-2t_2+2s_2+2q_2-2p_2} K^{q_1+q_2}\overline{K}^{p_1+p_2} H^{t_1+t_2}\overline{H}^{s_1+s_2} F) \\
&= -K^{j_1+j_2+q_1+q_2}\overline{K}^{p_1+p_2+i_1+i_2} H^{t_1+t_2+n_1+n_2}\overline{H}^{s_1+s_2+m_1+m_2}(EF - FE) \\
&= aP^{i_2+j_2+p_2+q_2} Q^{m_2+n_2+s_2+t_2} K^{j_1+q_1}\overline{K}^{i_1+p_1} H^{n_1+t_1}\overline{H}^{m_1+s_1} \\
&\quad -aP^{i_1+j_1+p_1+q_1} Q^{m_1+n_1+s_1+t_1} K^{j_2+q_2}\overline{K}^{i_2+p_2} H^{n_2+t_2}\overline{H}^{m_2+s_2}.
\end{aligned}$$

So, in order to get that $T(F)T(E) - T(E)T(F) = T(f(K, \overline{K}, H, \overline{H}))$, the following equality should be satisfied:

$$\begin{aligned}
& aP^{i_2+j_2+p_2+q_2} Q^{m_2+n_2+s_2+t_2} K^{j_1+q_1}\overline{K}^{i_1+p_1} H^{n_1+t_1}\overline{H}^{m_1+s_1} \\
&\quad -aP^{i_1+j_1+p_1+q_1} Q^{m_1+n_1+s_1+t_1} K^{j_2+q_2}\overline{K}^{i_2+p_2} H^{n_2+t_2}\overline{H}^{m_2+s_2} \\
&= aK^{j_1+q_1}\overline{K}^{i_1+p_1} H^{n_1+t_1}\overline{H}^{m_1+s_1} - aK^{j_2+q_2}\overline{K}^{i_2+p_2} H^{n_2+t_2}\overline{H}^{m_2+s_2}.
\end{aligned}$$

By Theorem 2.11, we can give a sufficient and necessary condition for this equality satisfied, i.e. $i_2 = j_2 = p_2 = q_2 = 0$ holds if and only if $i_1 = j_1 = p_1 = q_1 = 0$ holds, and $t_2 = n_2 = s_2 = m_2 = 0$ holds if and only if $t_1 = n_1 = s_1 = m_1 = 0$ holds.

Next, we have to verify that for any $x \in \mathfrak{w}_2 U_q$, $(id * T * id)(x) = x$. We only need to prove that $(id*T*id)(xy) = xy$, provided that $(id*T*id)(x) = x$, and $y$ is one of the generators $K, \overline{K}, H, \overline{H}, P, Q, E$ and $F$. Suppose that $(\Delta \otimes 1)\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$, then $(\Delta \otimes 1)\Delta(xK) = \sum x_{(1)}K \otimes x_{(2)}K \otimes x_{(3)}K$ and hence

$$(id * T * id)(xK) = \sum x_{(1)} K T(x_{(2)}K) x_{(3)} K = xK.$$

Similarly,

$$\begin{aligned}
(id * T * id)(xE) &= \sum x_{(1)} A_1 T(x_{(2)} A_1) x_{(3)} E + \sum x_{(1)} A_1 T(x_{(2)} E) x_{(3)} B_1 \\
&\quad + \sum x_{(1)} E T(x_{(2)} B_1) x_{(3)} B_1 \\
&= xE + \sum x_{(1)} A_1 T(E) T(x_{(2)}) x_{(3)} B_1 \\
&\quad + \sum x_{(1)} E T(B_1) T(x_{(2)}) x_{(3)} B_1 \\
&= xE.
\end{aligned}$$

Here, $A_1 = K^{i_1}\overline{K}^{j_1} H^{m_1}\overline{H}^{n_1}$, $B_1 = K^{i_2}\overline{K}^{j_2} H^{m_2}\overline{H}^{n_2}$. Similarly, we can prove that $(id * T * id)(xy) = xy$ holds when $y$ is one of the remaining generators. Using the similar method, we can prove $(T*id*T)(x) = T(x)$ for any $x \in \mathfrak{w}_2 U_q$.

The other relations which we have not checked are also satisfied. The proof can be seen that in Theorem 3.2.

Now, by Lemma 3.1, we have proven the result.

## References


1. Aizawa, N.; Isaac, P.S.: Weak Hopf algebras corresponding to $\mathrm{U_q[sl_n]}$. J. Math. Phys. **44**, 5250–5267 (2003)
2. Bergman, G.: The diamond lemma for ring theory. Adv. Math. **29**(2), 178–218 (1978)
3. Block, R.E.: The irreducible representations of the Lie algebra $\mathfrak{sl}(2)$ and of the Weyl algebra. Adv. Math. **39**, 69–110 (1981)
4. Böhm, G.; Korinél, S.: A coassociative $C^*$-quantum group with nonintegral dimensions. Lett. Math. Phys. **38**(4), 437–456 (1996)
5. Böhm, G.; Nill, F.; Korinél, S.: Weak Hopf algebras I. Integral theory and $C^*$ structure. J. Algebra **221**, 385–438 (1999)
6. Drinfeld, V.G.: Quantum groups. In: Proc. Int. Cong. Math. (Berkeley, 1986), pp. 798–820. Amer. Math. Soc., Providence (1987)
7. Duplij, S.; Sinelshchikov, S.: Quantum enveloping algebras with von Neumann regular Cartanlike generators and the Pierce decomposition. Comm. Math. Phys. **287**, 769–785 (2009)
8. Green, J.A.; Nicols, W.D.; Taft, E.J.: Left Hopf algebras. J. Algebra **65**, 399–411 (1980)
9. Hayashi, T.: Quantum group symmetry of partition functions of IRF models and its application to Jones' index theory. Commun. Math. Phys. **157**(2), 331–345 (1993)
10. Ji, Q.Z.; Wang, D.G.: Finite-dimensional representations of quantum groups $\mathrm{U_q(f(K))}$. East-West J. Math. **2**(2), 201–213 (2000)
11. Jimbo, M.: A q-difference analogue of $U(g)$ and the Yang-Baxter equation. Lett. Math. Phys. **10**, 63–69 (1985)
12. Kassel, C.: Quantum Groups. Springer, New York (1995)
13. Li, F.: Weak Hopf algebras and some new solutions of the quantum Yang-Baxter equation. J. Algebra **208**, 72–100 (1998)
14. Li, F.: Weak Hopf algebras and regular monoids. J. Math. Res. Exposition **19**, 325–331 (1999)
15. Li, F.; Duplij, S.: Weak Hopf algebras and singular solutions of quantum Yang-Baxter equation. Commun. Math. Phys. **225**, 191–217 (2002)
16. Madore, J.: Introduction to Noncommutative Geometry and its Applications. Cambridge University Press, Cambridge (1995)
17. Montgomery, S.: Hopf Algebras and Their Action on Rings. CBMS, Lecture in Math., vol. 82. AMS, Providence (1993)
18. Ocneanu, A.: Quantized groups, string algebras and Galois theory for algebras. In: Operator Algebras and Applications, vol. 2, pp. 119–172. London Math. Soc. Lecture Note Ser., 136. Cambridge University Press, Cambridge (1988)
19. Ringel, C.M.: Tame Algebras and Integral Quadratic Forms. LNM 1099. Springer, Berlin (1984)
20. Sweedler, M.E.: Hopf Algebras. Benjamin, New York (1969)
21. Wang, D.G.; Ji, Q.Z.; Yang, S.L.: Finite-dimensional representations of quantum groups Uq(f(K)). Commun. Algebra **30**, 2191–2211 (2002)
22. Wu, Z.X.: A class of weak Hopf algebras related to a Borcherds-Cartan matrix. J. Phys. A. **39**, 14611–14626 (2006)
23. Yang, S.L.: Weak Hopf algebras corresponding to Cartan matrices. J. Math. Phys. **46**(7), Article ID 073502 (2005)
24. Ye, L.X.; Li, F.: Weak quantum Borcherds superalgebras and their representations. J. Math. Phys. **48**, Article ID 023502 (2007)